Chronotaxic dynamics: when the characteristic frequencies fluctuate and the system is stable

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I. INTRODUCTION

Complex, fluctuating dynamics abounds in nature. With the fast sensors available today, large numbers of time-series are continuously being measured and analysed. To characterise the underlying dynamics one needs models. Although there are no strict boundaries, two general approaches coexist: stochastic and deterministic, as illustrated in Fig. (1). Of course, the same time series can be characterised either as *stochastic* or as *deterministic*. In the stochastic approach we do not make any assumptions about the properties or values of parameters involved in generating the dynamical behaviour and we analyse the time series using probability theory. But often, in real problems, we need to know the reason for a change in behaviour, for example from a healthy state to a diseased one. In this case we try to generate hypotheses or models and describe the causal relationship using deterministic systems.

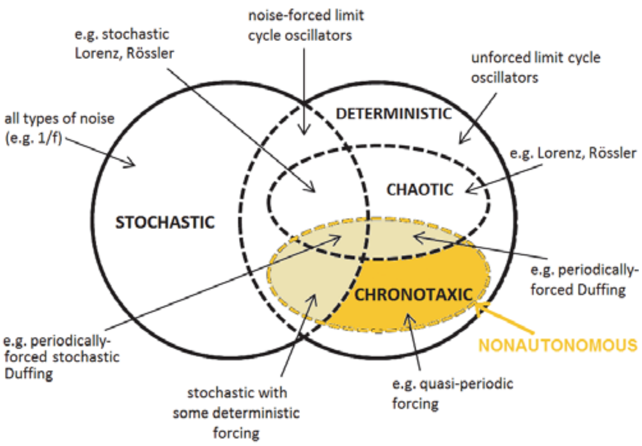

FIG. 1. A simplified representation of dynamical systems with typical examples. Although the chronotaxic systems are fully deterministic, unless influenced by stochastic perturbations, they have been usually treated as stochastic.

Generally, deterministic systems are considered within the theory of autonomous systems. In the last decades, much research has been devoted to chaotic systems, which are a special class of deterministic systems. Of course, as soon as they are perturbed by being coupled to a stochastic system their characteristics change as indicated in Fig. (1).

But many living systems, as well as man-made ones, are thermodynamically open because they exchange energy and matter with the environment. For such systems, the mathematical theory of non-autonomous systems is necessary. Motivated by these needs, recent developments have resulted in advances in non-autonomous dynamical systems theory[1], and in the theory of random dynamical systems[2].

Furthermore, only when a system is described as non-autonomous, and not transformed into autonomous by adding an extra dimension to account for the time-dependence, can one explain the *stability of its time-dependent dynamics*: such stability does not allow the time-variable dynamics to be changed easily by external *continuous* perturbation, a feature that provides a foundation for the newly-established theory of chronotaxic systems[3–5]. The ability of such systems to sustain stability in the amplitude and phase of oscillations under continuous perturbation are key features that were used in naming them chronotaxic (from *chronos* – time and *taxis* – order). Chronotaxic systems possess a *time-dependent point attractor* provided by an external drive system. This allows the frequency of oscillations to be prescribed externally through this driver and response system, giving rise to determinism even when faced with strong perturbations.

It can be shown[5] that the existence of such stability can cause their non-autonomous dynamics to look stochastic-like, and very complex, perhaps leading to their misidentification as stochastic or chaotic. In fact, because the underlying deterministic dynamics of such systems remains stable, we can decompose the dynamics into a deterministic part and a component due to the external perturbations.

We will now briefly introduce the basic properties of chronotaxic systems and discuss some open questions.

II. CHRONOTAXIC SYSTEMS IN BRIEF

A chronotaxic system[3–5] is a non-autonomous oscillatory dynamical system $\mathbf{x}$ generated by an autonomous system of unidirectionally coupled equations

$$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}), \quad \dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{p}), \qquad (1)$$

where $\mathbf{p} \in R^n$, $\mathbf{x} \in R^m$, $\mathbf{f} : R^n \to R^n$, $\mathbf{g} : R^m \times R^n \to R^m$; n and m can be any positive integers. The system (1) may also be called a drive and response system[6], or a master-slave configuration[7]. Considered in the context of non-autonomous dynamical systems[1,8], the system (1) can be viewed as a skew-product flow or as a process.

Importantly, the solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$ of Eqs. (1), depends on the actual time t as well as on the initial conditions $(t_0, \mathbf{x}_0)$; whereas the solution $\mathbf{p}(t - t_0, \mathbf{p}_0)$ depends only on initial condition p_0 and on the time of evolution $t - t_0$. The subsystem $\mathbf{x}$ is nonautonomous in the sense that it can be described by an equation which depends on time explicitly, e.g. $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{p}(t))$. A chronotaxic system is described by $\mathbf{x}$ which is assumed to be observable, and $\mathbf{p}$ which may be inaccessible for observation, as often occurs when studying real systems. Rather than assuming or approximating the dynamics of $\mathbf{p}$; we focus on the dynamics of $\mathbf{x}$ and use only the following simple assumption: that system $\mathbf{p}$ is such that it creates a time-dependent steady state in the dynamics of $\mathbf{x}$.

Therefore, the whole external environment with respect to $\mathbf{x}$ is divided into two parts. The first part is given by $\mathbf{p}$ which is the part that makes the system $\mathbf{x}$ chronotaxic. The second part contains the rest of the environment and is therefore considered as external perturbations. The theory for the case where amplitudes and phases are separable have been introduced by Suprunenko *et al*[3,4], and it has subsequently been expanded to include the generalized case of chronotaxic systems where such decoupling is not required[5].

Thus, when perturbations do not destroy the chronotaxic properties of a system, the stable deterministic component of its dynamics can be identified, as shown by Clemson *et al*[9]. This reduces the complexity of the system, enabling us to filter out the stochastic component and focus on the deterministic dynamics and the interactions between system $\mathbf{x}$ and its driver $\mathbf{p}$, as shown in[3,9]. For complex and open systems it has the potential to extract properties of the system which were previously neglected.

III. OPEN PROBLEMS

The theory of chronotaxic systems could facilitate more realistic insight into the underlying dynamics of systems whose time-evolution is recorded. As chronotaxic systems are inevitably non-autonomous, inverse approach methods, developed for analysis of time-series with time-dependent characteristics[10], can safely be applied to gain initial insights. Specific methods have already been proposed for the detection of chronotaxicity[9,11]. However, they are applicable to systems in which the amplitude and phase dynamics are separable, as they are applied directly to the extracted phases of the system, while the amplitude information is discarded. Such an approach is valid given that the amplitude dynamics of a chronotaxic system corresponds to the convergence of the system to a limit cycle, influenced only by a negative Lyapunov exponent and external perturbations, while the phase dynamics corresponds to convergence to a time-dependent point attractor[9,11]. As it is the point attractor in the phase dynamics in which we are primarily interested, the separation into amplitude and phase follows naturally. Using this approach, an example of chronotaxic dynamics has already been demonstrated in a real system, in the case of heart rate variability under paced respiration[9].

However, in generalized chronotaxic systems[5], the amplitude and phase are not required to be separable, providing an even greater applicability to real systems, allowing for amplitude-amplitude and amplitude-phase interactions, in addition to the phase-phase dynamics considered hitherto[3,4]. Therefore, the incorporation of the ability to identify these new possibilities for chronotaxicity will be crucial to the further development of inverse approach methods. These will then provide the means to detect chronotaxicity in systems where the amplitude and phase are not separable, as e.g. is the case of brain dynamics. However, this is an open problem, as yet awaiting a solution.

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