Rates of rare events: scaling, fragility, and delay effects

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I. INTRODUCTION

We consider the tails of the probability distribution and the escape rates in classical and quantum systems away from thermal equilibrium. These characteristics and the very mechanisms of the appropriate large rare fluctuations display features, which have no analogs in equilibrium systems. An example is switching between coexisting stable periodic states due to quantum fluctuations via the mechanism, which we call quantum activation and which is neither tunneling nor classical activation.[1] Despite the fact that there are no known general principles, which would describe the rates of rare events, these rates display some universal features. In particular, they display scaling behavior near bifurcation points where the number of the stable states changes. We will describe this behavior and provide a comparison with the experiment. We will also discuss the effect of fragility of the switching rates. This effect consists in the nonanalytic dependence of the logarithms of the rates on the system parameters in the relevant limit. It has been found in such diverse problems as population dynamics and nonequilibrium quantum fluctuations. However, the general nature of the fragility remains elusive, this is an unsolved problem of noise. We will also outline the results on rare fluctuations in classical systems with delayed dissipation. In such systems, the most probable paths followed in fluctuations to a given state or in switching are given by acausal equations. This leads to nontrivial consequences and a strong modification of the rates even for small delay.

II. QUANTUM ACTIVATION

A reliable platform for studying quantum phenomena away from thermal equilibrium is provided by resonantly modulated vibrational systems, such as modes of superconducting cavities or plasma vibrations of Josephson junctions. These modes (oscillators) have a small decay rate Γ compared to the eigenfrequency ω_0 and weak nonlinearity. When driven close to ω_0 or parametrically modulated close to $2\omega_0$, they display almost sinusoidal vibrations. However, when the driving amplitude F is not too small, because of the interplay between the weak nonlinearity and slow decay, they can have several coexisting vibrational states.

The coupling of a driven oscillator to a thermal reservoir leads to quantum noise. This noise and the motion of the oscillator as a whole can be conveniently described in the rotating wave approximation (RWA). For fairly general assumptions about the coupling, on times slow compared to the vibration period $2\pi/\omega_0$ one can derive a quantum Langevin equation for the scaled dimensionless coordinate Q and momentum P in the rotating frame. It can be written as

$$\dot{Q} = -i\tilde{\hbar}^{-1}[Q,\hat{g}] - Q + \hat{f}_Q(\tau),$$
$$\dot{P} = -i\tilde{\hbar}^{-1}[P,\hat{g}] - P + \hat{f}_P(\tau). \quad (1)$$

Here, $\tilde{\hbar}$ is the scaled dimensionless Planck constant and $\hat{g}$ is the Hamiltonian of the driven oscillator in the rotating frame. It does not depend on time in the RWA. The explicit form of $\tilde{\hbar}$ and $\hat{g}$ is given in Ref. 1. The eigenvalues of the operator $\hat{g}$ give the scaled quasienergy of the periodically driven oscillator.

In Eq. (1), $\hat{f}_{Q,P}$ are quantum noise operators. The noise is δ-correlated in slow time,

$$\langle \hat{f}_Q(\tau)\hat{f}_Q(\tau')\rangle = \langle \hat{f}_P(\tau)\hat{f}_P(\tau')\rangle = \tilde{\hbar}(2\bar{n}+1)\delta(\tau-\tau'),$$
$$\langle[\hat{f}_Q(\tau),\hat{f}_P(\tau')]\rangle = 2i\tilde{\hbar}\delta(\tau-\tau'); \quad (2)$$

$\bar{n} = [\exp(\hbar\omega_0/k_BT)-1]^{-1}$ is the oscillator Planck number.

Equations (1) simplify for the values of the driving field F and its frequency ω_F close to the bifurcation points where the number of the stable states changes. Here, one of the motions in the system becomes slow, there emerges a soft mode with coordinate $\tilde{Q}$, which is a linear combination of Q and P. The dynamics of this mode is essentially classical, since it commutes with itself, with equation of motion of the form $\dot{\tilde{Q}} = -\partial_{\tilde{Q}}U(\tilde{Q}) + f_{\tilde{Q}}(\tau)$. The only signature of the quantumness is the noise intensity $f_{\tilde{Q}}$, which is given by Eq. (2) and explicitly contains $\hbar$, whereas the form of $U(\tilde{Q})$ is determined by the type of the bifurcation point. The problem of switching is then mapped on the Kramers problem of escape of an overdamped particle. The escape rate is

$$W_{\rm sw} = \Omega_{\rm sw}\exp(-R_A/\tilde{\hbar}). \quad (3)$$

The quantum activation energy is $R_A = \Delta U/(\bar{n}+1/2)$, where ΔU is the height of the barrier to be overcome in escape; $\Omega_{\rm sw}$ is the Kramers prefactor. Both for the saddle-node and pitchfork bifurcation points, which are of interest for driven oscillators, R_A and $\Omega_{\rm sw}$ scale as powers of the distance to the bifurcation point.[1]

Of significant interest for the current experiments on parametrically driven quantum oscillators is the vicinity of the critical point $F = F_c, \omega_F = 2\omega_0$ where there first emerge period-two vibrations for parametric driving. The dynamics in this parameter range is significantly different from that near simple (co-dimension one) bifurcation points. We have now developed a method to analyze

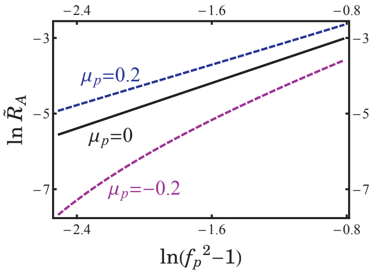

FIG. 1. Scaling of the quantum activation energy $\tilde{R}_A = (\bar{n} + 1/2)R_A$ with the relative driving field amplitude $f_{\rm p} = F/F_c$ for transitions between the parametrically excited period-two vibrational states with opposite phases. In the shown range $1.04 < f_{\rm p} < 1.2$ the scaled frequency detuning $\mu_{\rm p} \propto (\omega_F - 2\omega_0)/\omega_F$ is such that the period-two vibrations are the only stable states of the oscillator.

this dynamics. There is no simple single scaling law that would describe the behavior of R_A. The results obtained for R_A are shown in Fig. 1.

Of significant interest is the parameter range where the motion of the oscillator in the rotating frame is underdamped. With no dissipation, the oscillator would stay in a state with given quasienergy or in a superposition of such states. Dissipation leads to drift over quasienergy toward its value in the classically stable vibrational state. However, it also leads to diffusion over quasienergy. Such diffusion occurs even for $T = 0$ and is of purely quantum origin. It results in a stationary distribution ρ_n over quasienergy states $|n\rangle$ of the Boltzmann type, $\rho_n \propto \exp[-R_n/\tilde{\hbar}]$. For small $\hbar$ one has $R_n \equiv \int_{g_{st}}^{g_n} dg R'(g)$, where $g_n = \langle n|\hat{g}|n\rangle$ and g_{st} is the value of $\hat{g}$ at the stable state. Function $R'(g)$ plays the role of the reciprocal effective temperature, which depends on g.

Remarkably, the small-$\tilde{\hbar}$ approximation breaks down where $T \to 0$. The value of R' obtained for $T = 0$ and $T \to 0$ are different. We have recently shown[2] that the quasi-Boltzmann form of the distribution does not apply in a narrow temperature range $T_{c1} < T < T_{c2}$, where the critical temperatures are $T_{c1} \propto \hbar^2$ and $T_{c2} \propto \hbar/|\log \tilde{\hbar}|$. One can say that $R'(g)$ has a kink. Such kink is shown in Fig. 2.[2]

We will discuss the general conditions for the onset of fragility and provide an analytical theory of the kink of $R'(g)$. We will also compare the onset of fragility in a quantum system and in several models of population dynamics. So far the effect has been seen in systems with discrete states, like quasienergy states or the numbers of species. An important open question is whether fragility also emerges in continuous systems.

III. SWITCHING IN SYSTEMS WITH DELAY

Delay naturally arises in dissipative dynamical systems. In such systems, dissipation results from the coupling to a reservoir: motion of the system causes changes in the reservoir, which in turn affect the motion. The underlying reaction of the reservoir is generically delayed. Along with the dissipative force, the reservoir exerts a random force on the system. If dissipation is delayed, the random force has a finite correlation time.

We will discuss our recent results[3] on the probability distribution and switching rates in systems with delay. They are related to systems where chaotic motion in the absence of noise does not play a role. There are several key elements here. First, even though to find a dynamical trajectory without noise one has to know the whole dynamical history of the system, the trajectory, which is most likely to be followed in a large fluctuation to a given state, is well defined. It is well-defined also for the problem of escape from a metastable state. The "price" for having delay is that the functional, which we obtain and which determines the distribution of fluctuational trajectories, is nonlocal in time. Therefore the equations for the most probable trajectories, which are the extremals of this functional, are acausal.

We will present explicit results for the switching rates close to bifurcation points of the system, and also for the case of small delay compared to the system relaxation time. The effect of the finite correlation time of the noise will be also discussed.

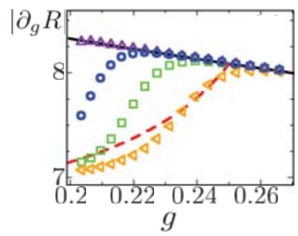

FIG. 2. Effective reciprocal temperature $|\partial_g R|$ calculated numerically from $\ln(\rho_n/\rho_{n-1})$. The results refer to three temperatures in the range $T_{c1} < T < T_{c2}$. The total number of localized quasienergy states in the basin of attraction of the small-amplitude state of a resonantly driven oscillator is $M = 20$. The solid and dashed lines show $|\partial_g R|$ for $\bar{n} = 0$ and for $\bar{n} \to 0$ but $T > T_{c2}$, respectively.[2]

[1] M. I. Dykman, in *Fluctuating Nonlinear Oscillators: from Nanomechanics to Quantum Superconducting Circuits*, ed. by M.I. Dykman (OUP, Oxford, 2012), p. 165.

[2] L. Guo, V. Peano, M. Marthaler, and M. I. Dykman, Phys. Rev. A, **87**, 062117 (2013).

[3] M. I. Dykman and I. B. Schwartz, Phys. Rev. E **86**, 031145 (2012); I. B. Schwartz et al., *ibid.* **91**, 012139 (2015).