

Stationary states in 2D systems driven by Lévy noises

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Motivation: Boltzmann-Gibbs distribution

In the equilibrium

$$P(\text{state}) \propto \exp \left[-\frac{E}{k_B T} \right],$$

T – system temperature, E – energy of the state.

For an **overdamped** particle the Langevin equation is

$$\frac{dx}{dt} = -V'(x) + \sqrt{2k_B T} \xi(t).$$

Particle's energy is

$$E = V(x)$$

and the stationary distribution

$$P(x) \propto \exp \left[-\frac{V(x)}{k_B T} \right]$$

is fully determined by the potential $V(x)$.



Motivation

- Examination of stationary states for more general noises.

Road map of presentation

- basic definitions:
 - 1D α -stable noises,
 - 2D α -stable noises.
- stationary states for 1D and 2D systems.

Try to understand

- role of increasing spatial dimensionality,
- universalities of noise driven systems.

Take home message

2D α -stable noises differs from their 1D analogs but systems driven by 2D α stable noises display universal properties.



A random variable X is stable if

$$AX^{(1)} + BX^{(2)} \stackrel{d}{=} CX + D,$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies of X , $\stackrel{d}{=}$ denotes equality in distributions. The random variable X is called strictly stable if $D = 0$. The random variable X is symmetric stable if it is stable and

$$\text{Prob}\{X\} = \text{Prob}\{-X\}.$$

The random variable is α -stable if $C = (A^\alpha + B^\alpha)^{1/\alpha}$ where $0 < \alpha \leq 2$.

The characteristic function of α -stable densities is

$$\phi(k) = \mathbb{E} \left[e^{ikX} \right] = \begin{cases} \exp \left[-\sigma^\alpha |k|^\alpha \left(1 - i\beta \text{sign} k \tan \frac{\pi\alpha}{2} \right) + i\mu k \right] & \text{if } \alpha \neq 1, \\ \exp \left[-\sigma |k| \left(1 + i\beta \frac{2}{\pi} \text{sign} k \ln |k| \right) + i\mu k \right] & \text{if } \alpha = 1, \end{cases}$$

where $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\sigma > 0$ and $\mu \in \mathbb{R}$.



Characteristic function

$$\phi(k) = \begin{cases} \exp \left[-\sigma^\alpha |k|^\alpha \left(1 - i\beta \operatorname{sign} k \tan \frac{\pi\alpha}{2} \right) + i\mu k \right], & \text{for } \alpha \neq 1, \\ \exp \left[-\sigma |k| \left(1 + i\beta \frac{2}{\pi} \operatorname{sign} k \ln |k| \right) + i\mu k \right], & \text{for } \alpha = 1, \end{cases}$$

- asymptotic behavior $P(x) \propto |x|^{-(\alpha+1)}$ ($\alpha < 2$),
- Normal distribution ($\alpha = 2, \beta = 0$)

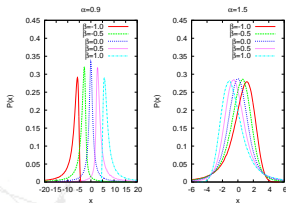
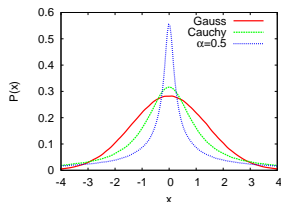
$$\frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(x - \mu)^2}{2\sigma^2} \right],$$

- Cauchy distribution ($\alpha = 1, \beta = 0$)

$$\frac{\sigma}{\pi} \frac{1}{(x - \mu)^2 + \sigma^2},$$

- Lévy-Smirnoff distribution (fully asymmetric, $\alpha = \frac{1}{2}, \beta = 1$)

$$\left(\frac{\sigma}{2\pi} \right)^{\frac{1}{2}} (x - \mu)^{-\frac{3}{2}} \exp \left[-\frac{\sigma}{2(x - \mu)} \right].$$



Analogously like in 1D: Random vector $\mathbf{X} = (X_1, \dots, X_d)$ is said to be a stable random vector in \mathbb{R}^d if for any positive numbers A and B , there is a positive number C and a vector \mathbf{D} such that

$$A\mathbf{X}^{(1)} + B\mathbf{X}^{(2)} \stackrel{d}{=} C\mathbf{X} + \mathbf{D},$$

where $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent copies of \mathbf{X} , $\stackrel{d}{=}$ denotes equality in distributions. The vector \mathbf{X} is called strictly stable if $\mathbf{D} = \mathbf{0}$. The vector \mathbf{X} is symmetric stable if it is stable and

$$\text{Prob}\{\mathbf{X} \in A\} = \text{Prob}\{-\mathbf{X} \in A\}$$

for any Borel set A of \mathbb{R}^d . A random vector is α -stable if $C = (A^\alpha + B^\alpha)^{1/\alpha}$ where $0 < \alpha \leq 2$.



The characteristic function $\phi(\mathbf{k}) = \mathbb{E} [e^{i\langle \mathbf{k}, \mathbf{X} \rangle}]$ of the α -stable vector $\mathbf{X} = (X_1, \dots, X_d)$ in \mathbb{R}^d is

$$\phi(\mathbf{k}) = \begin{cases} \exp \left\{ - \int_{S_d} |\langle \mathbf{k}, \mathbf{s} \rangle|^\alpha \left[1 - i \operatorname{sign}(\langle \mathbf{k}, \mathbf{s} \rangle) \tan \frac{\pi\alpha}{2} \right] \Gamma(d\mathbf{s}) + i \langle \mathbf{k}, \boldsymbol{\mu}^0 \rangle \right\} \\ \text{for } \alpha \neq 1, \\ \exp \left\{ - \int_{S_d} |\langle \mathbf{k}, \mathbf{s} \rangle|^\alpha \left[1 + i \frac{2}{\pi} \operatorname{sign}(\langle \mathbf{k}, \mathbf{s} \rangle) \ln(\langle \mathbf{k}, \mathbf{s} \rangle) \right] \Gamma(d\mathbf{s}) + i \langle \mathbf{k}, \boldsymbol{\mu}^0 \rangle \right\} \\ \text{for } \alpha = 1, \end{cases}$$

where S_d is a unit sphere in \mathbb{R}^d and $\Gamma(\cdot)$ is a spectral measure.

G. Samorodnitsky, and M. S. Taqqu, *Stable NonGaussian Random Processes*, (Chapman & Hall 1994).



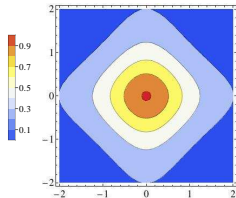
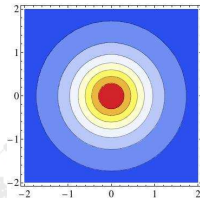
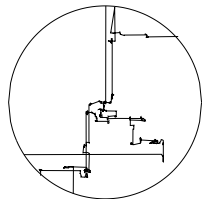
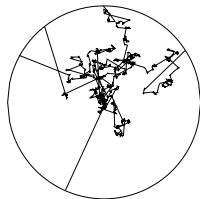
Cauchy distribution $\alpha=1$

For symmetric spectral measure concentrated on intersections of the axes with the unit sphere S_2 the bi-variate Cauchy ($\alpha = 1$) distribution is

$$p(x, y) = \frac{1}{\pi} \frac{\sigma}{(x^2 + \sigma^2)} \times \frac{1}{\pi} \frac{\sigma}{(y^2 + \sigma^2)}.$$

For continuous and uniform spectral measure

$$p(x, y) = \frac{1}{2\pi} \frac{\sigma}{(x^2 + y^2 + \sigma^2)^{3/2}}.$$



The Langevin equation

$$\frac{dx}{dt} = -V'(x) + \sigma\zeta_{\alpha,0}(t),$$

$$dx = -V'(x)dt + \sigma dL_{\alpha,0}(t)$$

is associated with the fractional Smoluchowski-Fokker-Planck equation

$$\begin{aligned}\frac{\partial p(x, t)}{\partial t} &= \frac{\partial}{\partial x} [V'(x)p(x, t)] + \sigma^\alpha \frac{\partial^\alpha p(x, t)}{\partial |x|^\alpha} \\ &= \frac{\partial}{\partial x} [V'(x)p(x, t)] - \sigma^\alpha (-\Delta)^{\alpha/2} p(x, t).\end{aligned}$$

The fractional Riesz-Weil derivative is defined via its Fourier transform

$$\mathcal{F} \left[\frac{\partial^\alpha p(x, t)}{\partial |x|^\alpha} \right] = \mathcal{F} \left[-(-\Delta)^{\alpha/2} p(x, t) \right] = -|k|^\alpha \mathcal{F} [p(x, t)].$$

P. D. Ditlevsen, Phys. Rev. E **60** 172 (1999).

D. Schertzer and M. Larchevêque, J. Duan, V. V. Yanowsky, S. Lovejoy, J. Math. Phys. **42** 200 (2001).



For $\alpha < 2$, and $V(x) = |x|^c$ stationary states exist for $c > 2 - \alpha$.
Stationary states (if exist) have power-law asymptotics

$$\rho_{st}(x) \propto |x|^{-(c+\alpha-1)}.$$

For $c = 2$ the stationary density is the same as the stable distribution associated with the underlying noise.

For $V(x) = \frac{1}{4}x^4$ and $\alpha = 1$

$$\rho_{st}(x) = \frac{\sigma}{\pi(\sigma^{4/3} - \sigma^{2/3}x^2 + x^4)}.$$

- A. V. Chechkin, J. Klafter, V. Yu. Gonchar, R. Metzler and L. V. Tanatarov, Chem. Phys. **284** 233 (2002);
Phys. Rev. E **67**, 010102 (2003).
B. Dybiec, I. M. Sokolov, A. V. Chechkin, J. Stat. Mech. P07008 (2010).



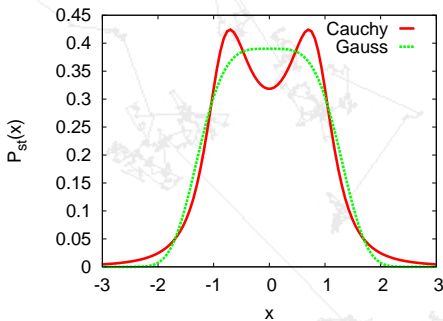
Stationary states (quartic – $V(x) = x^4/4$ – potential)

For $\alpha = 2$, the stationary states are of the Boltzmann-Gibbs type, i.e. $P(x) \propto \exp[-V(x)]$.

$$P_2(x) = \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \exp\left[-\frac{x^4}{4}\right].$$

For $\alpha < 2$, stationary solutions are no longer of the Boltzmann-Gibbs type. For $\alpha = 1$

$$P_1(x) = \frac{1}{\pi(x^4 - x^2 + 1)}.$$



A. V. Chechkin, J. Klafter, V. Yu. Gonchar, R. Metzler and L. V. Tanatarov, Chem. Phys. **284** 233 (2002); Phys. Rev. E **67**, 010102 (2003).



2D Langevin equation

$$\frac{d\mathbf{r}}{dt} = -\nabla V(\mathbf{r}) + \sigma\zeta_{\alpha}(t),$$

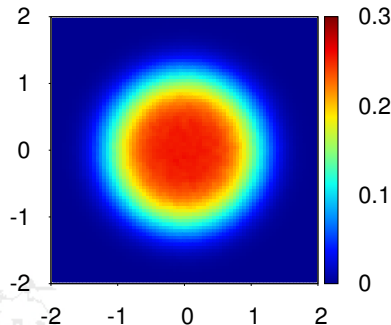
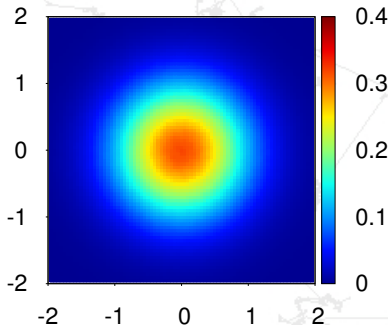
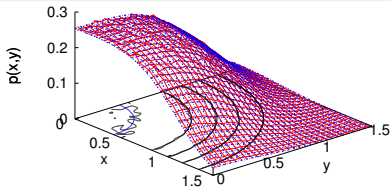
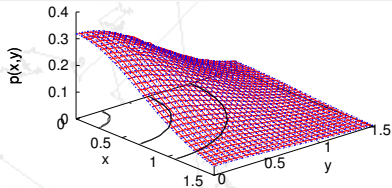
$$d\mathbf{r} = -\nabla V(\mathbf{r})dt + \sigma d\mathbf{L}_{\alpha}(t).$$

Especially interesting potentials are

- harmonic: $V(x, y) = \frac{1}{2}r^2 = \frac{1}{2}(x^2 + y^2)$,
- quartic: $V(x, y) = \frac{1}{4}r^4 = \frac{1}{4}(x^2 + y^2)^2$.



Bivariate Gaussian



$V(x, y) = \frac{1}{2}(x^2 + y^2)$ (left panel) and $V(x, y) = \frac{1}{4}(x^2 + y^2)^2$ (right panel) subject to the bi-variate, uniform Gaussian white noise ($\alpha = 2$).



The associated Smoluchowski-Fokker-Planck equation

$$\frac{\partial p(\mathbf{r}, t)}{\partial t} = \nabla \cdot [\nabla V(\mathbf{r})p(\mathbf{r}, t)] + \sigma^\alpha \Xi p(\mathbf{r}, t),$$

where Ξ is the fractional operator. $\nabla \cdot [\nabla V(\mathbf{r})p(\mathbf{r}, t)]$ originates due to the deterministic force $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$ acting on a test particle.

For the bi-variate α -stable noise with the uniform spectral measure the fractional operator

$$\Xi = -(-\Delta)^{\alpha/2}.$$

For the bi-variate α -stable noise with the discrete symmetric spectral measure (located on intersections of S_2 with axis)

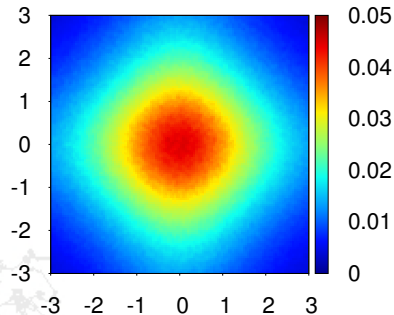
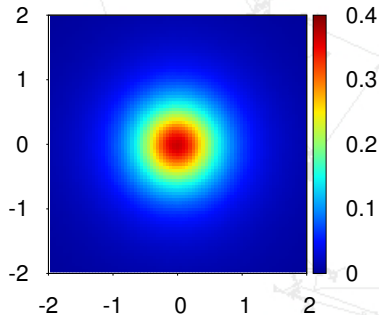
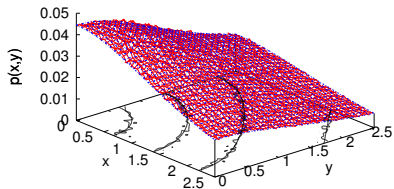
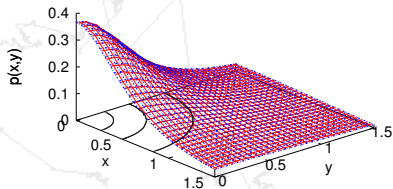
$$\Xi = \frac{\partial^\alpha}{\partial |x|^\alpha} + \frac{\partial^\alpha}{\partial |y|^\alpha}.$$

S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications* (Gordon and Breach, Yverdon, 1993).

A. V. Chechkin, V. Y. Gonchar, and M. Szydlowski, *Phys. Plasmas* **9**, 78 (2002).

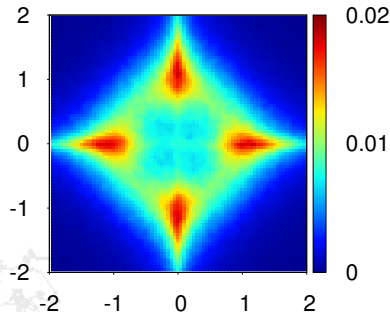
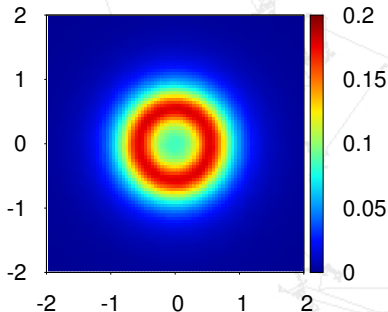
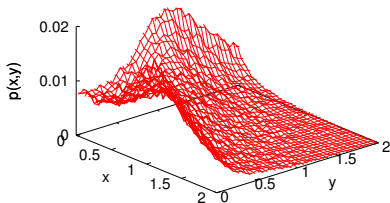
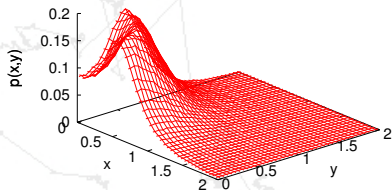


Bivariate Cauchy – parabolic potential



$V(x, y) = \frac{1}{2}(x^2 + y^2)$ with $\alpha = 1$ (Cauchy noise).

$\alpha = 0.5$ – quartic potential

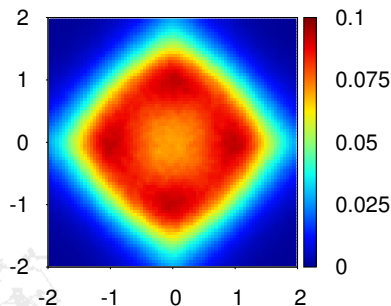
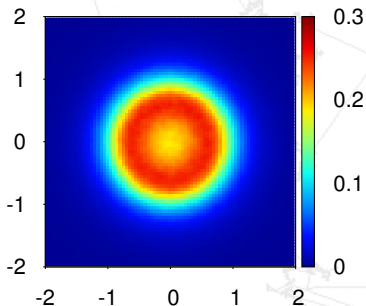
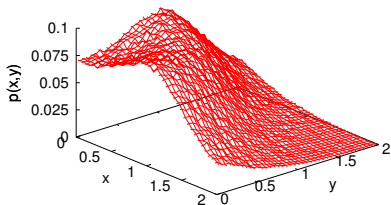
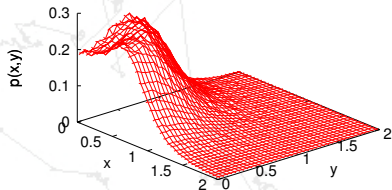


$$V(x, y) = \frac{1}{4}(x^2 + y^2)^2 \text{ with } \alpha = 0.5.$$

K. Szczepaniec and B. Dybiec, Phys. Rev. E **90**, 032128 (2014).



Bivariate Cauchy – quartic potential

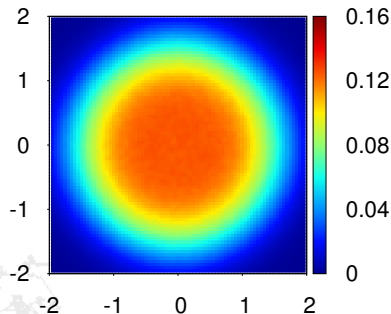
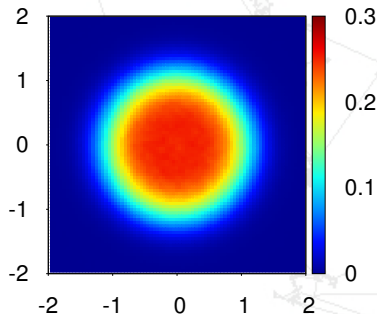
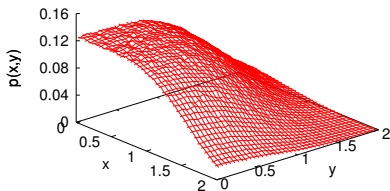
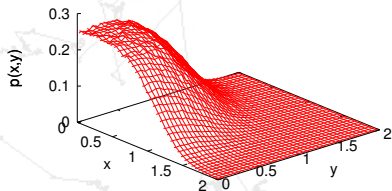


$$V(x, y) = \frac{1}{4}(x^2 + y^2)^2 \text{ with } \alpha = 1.$$

K. Szczepaniec and B. Dybiec, Phys. Rev. E **90**, 032128 (2014).



$\alpha = 1.9$ – quartic potential

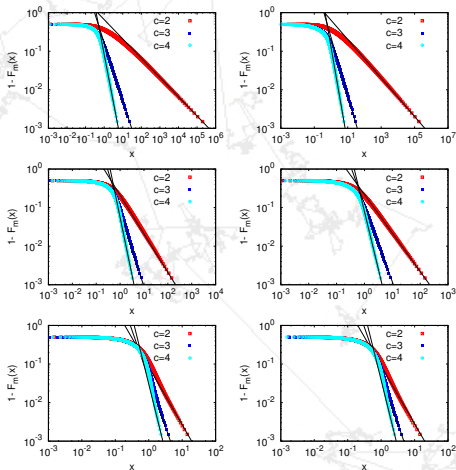


$$V(x, y) = \frac{1}{4}(x^2 + y^2)^2 \text{ with } \alpha = 1.9.$$

K. Szczepaniec and B. Dybiec, Phys. Rev. E **90**, 032128 (2014).



Marginal densities



Survival probabilities, $S(x) = 1 - F_m(x)$, for marginal densities of x for uniform (left panel) and symmetric discrete (right panel) spectral measures. α : $\alpha = 0.5$ (top row), $\alpha = 1$ (middle row) and $\alpha = 1.5$ (bottom row). Potentials of $V(x, y) = (x^2 + y^2)^{c/2}$ type: harmonic ($c = 2$), cubic ($c = 3$) and quartic ($c = 4$). Solid lines present $x^{-(c+\alpha-2)}$ power-law asymptotics of survival probabilities.

K. Szczepaniec and B. Dybiec, Phys. Rev. E **90**, 032128 (2014).



Conclusions

2D systems driven by bi-variate α -stable noises display analogous universalities like 1D systems.

Thank you very much for your attention!!

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