Non-hermitian Diffusion

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July 13th, 2015

UPON 2015, Barcelona

Supported in part by the grant DEC-2011/02/A/ST1/00119 of National Centre of Science.

Outline

- Why to bother about nonhermiticity
- Diffusion Redux
- Diffusion of hermitian matrices - "Dysonian way"
- Diffusion of hermitian matrices - "Burgulent way"
- Unraveling the diffusion of nonhermitian matrices
  [Burda, Grela, MAN, Tarnowski and Warchoł,
  Phys. Rev. Lett. 113 (2014) 104102,
  Nucl. Phys. B897 (2015) 421]
- Example
- Prospects and open problems

Nonhermitian operators

- Nonhermitian quantum mechanics (resonances, complex potentials...)
- Euclidean Quantum Field Theory (finite density QCD)
- Statistics (lagged correlators) $C_{i,j}(\Delta) = \frac{1}{T}\sum_{t=1}^{T} X_{i,t}X_{j,t+\Delta}$
- Complexity (directed graphs/networks, non-backtracking operators for sparse systems)

Diffusion Redux

- Wiener process $X_t = X_0 + B_t$, where
  $dB_t = B_{t+dt} - B_t = \mathcal{N}(0, dt)$
- $\frac{d}{dt}\langle F(B_t)\rangle = \frac{1}{2}\langle F_{xx}(B_t)\rangle$
  Proof: $F(B_{t+dt}) = F(B_t) + F_x(B_t)dB_t + \frac{1}{2}F_{xx}(B_t)dB_t^2$
  Diffusion : $< dB_t >= 0, \quad < dB_t^2 >= dt$
- Ito trick: $dF(B_t) \equiv F_x(B_t)dB_t + \frac{1}{2}F_{xx}(B_t)dt$
- Heat equation (Smoluchowski-Fokker-Planck eq.)
  $\partial_t \rho(x,t) = \frac{1}{2}\partial_{xx}\rho(x,t)$, where $< F(B_t) >\equiv \int F(x)\rho(x,t)dx$
- Example: For $\rho(x,0) = \delta(x)$, $\rho(x,t) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$

"Dysonian way" ([Dyson; 1962])

After considerable and fruitless efforts to develop a Newtonian theory of ensembles, we discovered that the correct procedure is quite different and much simpler...... from F.J. Dyson, J. Math. Phys. 3 (1962) 1192

- Wiener process: $H_\tau = H_0 + B_\tau$, where $\tau = \frac{t}{N}$
- Perturbation calculus for $H_{\tau+d\tau} = H_\tau + dH_\tau$ yields $\lambda_i(H_{\tau+d\tau}) = \lambda_i + <\psi_i|dH_\tau|\psi_i> + \frac{1}{2}\sum_{i\neq j}\frac{|<\psi_i|dH_\tau|\psi_j>|^2}{\lambda_i-\lambda_j}$
- Ito calculus: $d\lambda_i \equiv \frac{dB_i}{\sqrt{N}} + \frac{1}{N}\sum_{i\neq j}\frac{dt}{\lambda_i-\lambda_j}$
- Eigenvalues interact!

Diffusion of N by N hermitian matrices $H = H^\dagger$

- Gaussian Unitary Ensemble (GUE)

$$H_{ij} = \begin{cases} x_{ii} & \text{if } i = j \\ \frac{x_{ij}+iy_{ij}}{\sqrt{2}} & \text{if } i < j \end{cases}$$

where all x_{ij}, y_{ij} drawn from standard Gaussians, so

- $< dH_{ij} >= 0, \quad < (dH_{ij})^2 >= \frac{1}{N} dt$
- Probability distribution $\partial_t P(H,t) = LP(H,t)$, where $L = \frac{1}{2N}\sum_k \frac{\partial^2}{\partial^2 x_{kk}} + \frac{1}{2N}\sum_{i<j}\left(\frac{\partial^2}{\partial^2 x_{ij}} + \frac{\partial^2}{\partial^2 y_{ij}}\right)$
- $< F(H) >_t = \int [dH] P(H,t) F(H)$

"Burgulent way"

- We define $d_N(z,t) = \det(z\mathbf{1}_N - H)$
- Integrable, exact eq. (for any N and for any initial conditions)
  $\partial_t < d_N(z,t) >_t = -\frac{1}{2N}\partial_{zz} < d_N(z,t) >_t$
  [Blaizot,MAN,Warchoł; 2008-2013]
- Large N limit: $\lim_{N\to\infty} \frac{1}{N}\partial_z \ln < d_N > = \frac{1}{N}\partial_z < \ln d_N > = \frac{1}{N}\partial_z < \mathrm{tr} \ln(z\mathbf{1}_N - H) > \equiv g(z,t)$ (motivated by the Cole-Hopf transformation)
- Green's function $g(z,t) = \frac{1}{N}\left\langle \mathrm{tr} \frac{1}{z\mathbf{1}_N - H} \right\rangle = \frac{1}{N}\left\langle \sum_{k=1}^{N} \frac{1}{z-\lambda_k} \right\rangle$
- "Heat equation" becomes inviscid complex Burgers equation
  $\partial_t g + g\partial_z g = 0$ (case of Voiculescu eq. $\partial_t g + R(g)\partial_z g = 0$)
- Spectrum from Sochocki Plemelj eq.
  $\frac{1}{\lambda-\lambda' \pm i\epsilon} = \mathrm{P.V.}\frac{1}{\lambda-\lambda'} \mp i\pi\delta(\lambda-\lambda')$
- $\rho(\lambda,t) = -\frac{1}{\pi}\lim_{\epsilon\to 0} \Im g(z)|_{z=\lambda+i\epsilon}$
- Shock phenomena at the edges of the spectrum

"Burgelent way" - cont.

- "Eulerian" solution of Burgers equation (on complex plane) reads $g(z,t) = g_0(z - tg(z,t))$, so for simplest initial condition $H_0 = 0$, $g(z,0) = g_0(z) = \frac{1}{z}$, problem downgrades to the solution of the quadratic equation, i.e. reads $g(z,t) = \frac{1}{2t}(z - \sqrt{z^2 - 4t})$.
- Spectral density comes from the imaginary part of the Green's function, i.e. $\rho(\lambda,t) = \frac{1}{2\pi t}\sqrt{4t^2 - \lambda^2}$ (Diffusing Wigner's semicircle)
- Diffusing Wigner's semicircle (from Burgers equation) is a counterpart of the diffusing Gaussian (from the heat equation) in the world of large matrices.
- Finite N effects appear as a spectral viscosity $\nu_s \sim \frac{1}{2N}$, leading to universal spectral fluctuations in the vicinity of shock waves

Non-hermitian case - large N - electrostatic analogy

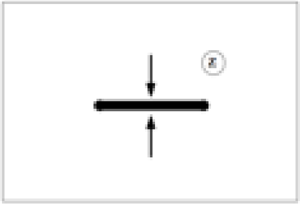
Analytic methods break down, since spectra are complex
$\rho(z,t) = \frac{1}{N}\left\langle \sum_i \delta^{(2)}(z-\lambda_i(t))\right\rangle$.

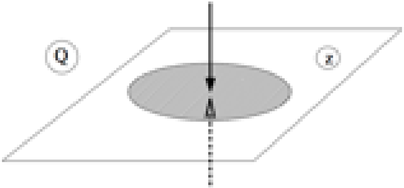
- Electrostatic potential
  $\phi(z,\bar{z},w,\bar{w},t) \equiv \lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\ln[|z-X|^2+|w|^2]\right\rangle$
  $= \lim_{N\to\infty}\left\langle \frac{1}{N}\ln D_N\right\rangle$ where
  $D_N(z,\bar{z},w,\bar{w}) = \det(Q\otimes \mathbf{1}_N - \mathcal{X})$ with
  $$Q = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \qquad \mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}$$
- Electric field $g = \partial_z \phi$
- Gauss law $\rho(z,t) = \frac{1}{\pi}\partial_{\bar{z}} g|_{w=0} = \frac{1}{\pi}\frac{\partial^2\phi}{\partial z\partial\bar{z}}|_{w=0}$
  Proof: $\delta^{(2)}(z) = \lim_{w\to 0}\frac{1}{\pi}\frac{|w|^2}{(|z|^2+|w|^2)^2}$

Hidden variable

Historically, $|w|$ was treated as an infinitesimal regulator only [Brown;1986],[Sommers et al.;1988]. We promote w to full, complex-valued dynamical variable.

Then, "orthogonal direction" w unravels the eigenvector correlator $O(z,t) = \frac{1}{\pi}\partial_w \phi \partial_{\bar{w}} \phi|_{w=0} = \frac{1}{N^2}\left\langle \sum_k O_{kk}\delta^{(2)}(z-\lambda_k(t))\right\rangle$, where $O_{ij} =< L_i|L_j >< R_j|R_i >$ and $|L_i >$ $(|R_i >)$ are left (right) eigenvectors of X.





Approach to nonhermitian matrices

- We supersede $d_N(z) = \det(z\mathbf{1}_N - H)$ by the determinant $D_N(z, \bar{z}, w, \bar{w}) = \det(Q \otimes \mathbf{1}_N - \mathcal{X})$
- For nonhermitian matrices X, we have left and right eigenvectors $X = \sum_k \lambda_k |R_k >< L_k| = \sum_k \bar{\lambda}_k |L_k >< R_k|$ where $X|R_k >= \lambda_k |R_k >$ and $< L_k|X = \lambda_k < L_k|$
- $< L_j|R_k >= \delta_{jk}$, but $< L_i|L_j > \neq 0$ and $< R_i|R_j > \neq 0$.
- $D_N = \det[U^{-1}(Q \otimes \mathbf{1}_N - \mathcal{X})U] =$ $\det \begin{pmatrix} z\mathbf{1}_N - \Lambda & -\bar{w} < L|L > \\ w < R|R > & \bar{z}\mathbf{1}_N - \bar{\Lambda} \end{pmatrix}$
- Spectrum (Λ) entangled with overlap of eigenvectors $O_{ij} \equiv < L_i|l_j >< R_j|R_i >$.

Random walk for the Ginibre ensemble

- We define random walk of $X_{ij} = x_{ij} + iy_{ij}$, where $dx_{ij} = \frac{1}{\sqrt{2N}} dB^x_{ij}$ and $dy_{ij} = \frac{1}{\sqrt{2N}} dB^y_{ij}$.
- We consider $< D_N(z, w, t) >= \int DX \ P(X, t) \det(Q - \mathcal{X})$
- Using Grassmannian integration tricks and the evolution equation $\partial_t P(X, \tau) = \frac{1}{4N} \sum(\partial^2_{x_{ij}} + \partial^2_{y_{ij}}) P(X, t)$ we arrive at exact 2d diffusion equation
  $\partial_t < D_N(z, w, t) >= \frac{1}{N} \partial_{w\bar{w}} < D_N(z, w, t) >$
- Solution reads $< D_N(z, |w|, t) >=$
  $\frac{2N}{t} \int_0^\infty q \exp\left(-N \frac{q^2 + |w|^2}{t}\right) I_0\left(\frac{2Nq|w|}{t}\right) D_N(z, q, t = 0) dq$
  where $D_N(z, |w|, t = 0) = \det((z - X_0)(\bar{z} - X_0^\dagger) + |w|^2)$.

"Burgulent way" - nonhermitian case, $N = \infty$ limit

- Let define $v \equiv |\partial_w \phi|$ and $|w| \equiv r$. Note that v^2 controls eigevectors and g controls the complex spectrum
- The hermitian-case Burgers equation $\partial_t g + g \partial_z g = 0$ is now superimposed by the system

$$\begin{aligned} \partial_t v &= v \partial_r v \\ \partial_t g &= \partial_z v^2 \end{aligned}$$

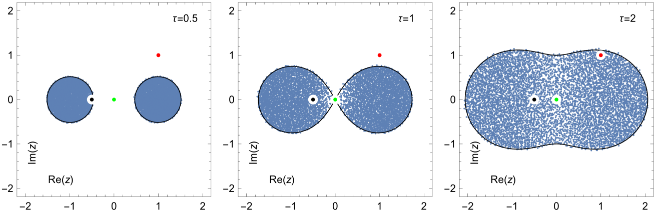
- Evolution of overlaps (v) **prior** to the evolution of spectra
- Shock phenomenon in eigenvector sector
- "Missed " complex plane (w) is relevant - quaternion (Q) description.

Examples

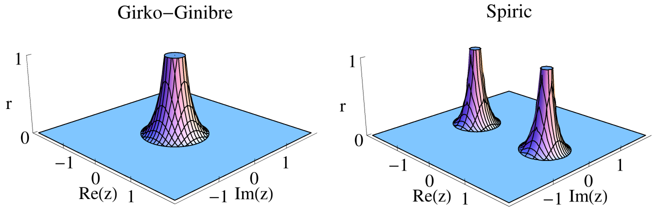
1. $X_0 = 0$
   $O(z,t) = \frac{1}{\pi t^2}(t - |z|^2)\Theta(\sqrt{t} - |z|)$
   [Chalker-Mehlig;1998],[Janik et al.;1998]
   $\rho(z,t) = \frac{1}{\pi t}\Theta(\sqrt{t} - |z|)$
   [Ginibre; 1964]
2. $X_0 = \mathrm{diag}(a, a, ..., -a, -a, ...)$

The spiric example

Initial condition $X_0 = \mathrm{diag}(a, ..a, -a, ... - a)$.

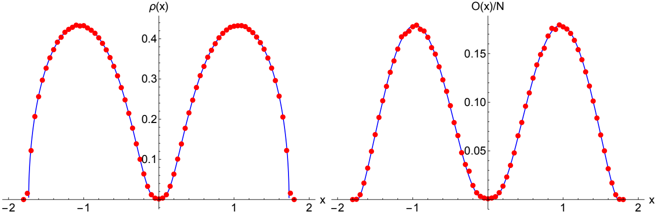


Evolution of singularities in (z, w) space. Ginibre versus spiric example



The spiric example cont.

Spectral density (left) and eigenvector correlator (right) snapshots



Conclusions and open problems

- Formalism of Dysonian dynamics for non-hermitian RMM, involving **coevolution of eigenvalues and eigenvectors**, for arbitrary N
- Conjecture, that above presented scenario, based on Ginibre ensemble, is generic for all non-hermitian RMM - **paramount role of eigenvectors**
- Unexpected **similarity** between hermitian and non-hermitian RMM based on **"Burgulence"** concepts
- Verification in various application of hermitian and non-hermitian random matrix models
- Unexplored mathematics

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