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The spectral characteristics of steady-state Lévy flights in an infinitely deep rectangular potential well

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Outline

- **Motivation**
- **The spectral and correlation characteristics: mathematic apparatus**
- **Brownian motion**
- **Anomalous diffusion in the form of Lévy flights**
- **Conclusions**

Motivation

Spectral densities of fluctuations provide an important tool to characterise physical systems, because they can be measured directly in experiments.

The investigations of spectra are useful to observe and analyse the interplay between fluctuations, relaxation and nonlinearity which are inherent to real physical systems. This interplay ranks among the most challenging problems of modern nonlinear physics and forms the basis of well-known nonlinear phenomena like

- **stochastic resonance;**
- **resonant activation;**
- **noise-enhanced stability;**
- **ratchet-effect, *etc.***

The spectral and correlation characteristics: mathematical basis

We start from the following general operator formula for the correlation function *K[τ]* of a stationary Markovian process *x(t)*

$$K[\tau] = \left\langle x\, e^{\hat{L}^{+}(x)\tau} x \right\rangle \qquad (1)$$

S.Yu. Medvedev, Radiophys. Quant. Electr. 20, 863 (1977).

where *L+(x)* is the adjoint kinetic operator.

According to the Wiener-Khinchin theorem the spectral power density reads

$$S(\omega) = \int_{-\infty}^{\infty} K[\tau] \cos\omega\tau d\tau = 2\mathrm{Re}\left\{\tilde{K}[i\omega]\right\} \qquad (2)$$

where *K[p]* is the Laplace transform of *K[τ]*.

Thus, from eq. (1) we arrive at

$$\tilde{K}[p] = \left\langle x \frac{1}{p - \hat{L}^{+}(x)} x \right\rangle \qquad (3)$$

The spectral and correlation characteristics: mathematical basis

One has to solve the following integro-differential equation for the auxiliary function *φ(x)*

$$\hat{L}^{+}(x)\varphi(x) - p\varphi(x) = -x \qquad (4)$$

and to find the average

$$\tilde{K}[p] = \langle x\varphi(x)\rangle \qquad (5)$$

In particular, the correlation time can be calculated as

$$\tau_k = \frac{1}{\langle x, x\rangle}\int_0^{\infty} K[\tau]d\tau = \frac{\tilde{K}(0)}{2\langle x, x\rangle} \qquad (6)$$

where <x, x> is the variance of random process *x(t).*

The spectral and correlation characteristics of Brownian diffusion

The Langevin equation

$$\frac{dx}{dt} = -U'(x) + \xi(t) \qquad (7)$$

where: *U(x)* is a potential and *ξ(t)* is white Gaussian noise with zero mean and the intensity *2D.*

The Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left[\frac{dU(x)}{dx}P\right] + D\frac{\partial^2 P}{\partial x^2} \qquad (8)$$

The adjoint kinetic operator

$$\hat{L}^{+}(x) = D\frac{\partial^2}{\partial x^2} - U'_x\frac{\partial}{\partial x} \qquad (9)$$

The steady-state probability distribution

$$P_{st}(x) = C_0 e^{-U(x)/D} \qquad (10)$$

The spectral and correlation characteristics of Brownian diffusion

Exact result for the correlation time (symmetric potential)

$$\tau_c = \frac{1}{D}\frac{\int_0^\infty e^{U(y)/D}\left[\int_y^\infty x\,e^{-U(x)/D}dx\right]^2 dy}{\int_0^\infty x^2 e^{-U(x)/D}dx} \quad (11)$$

H. Riskin (1989)

Piece-wise linear bistable potential

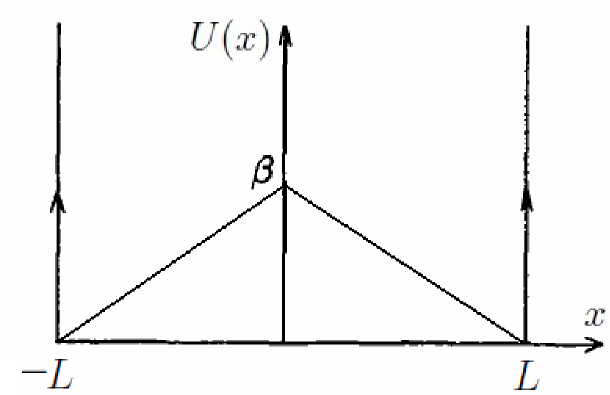
$$\tau_c = \frac{\theta}{\beta^2}\frac{(\beta-1)^2 e^{\beta} + (-\beta^3+3\beta^2-4\beta+4) - 5e^{-\beta}}{(\beta^2-2\beta+2)-2e^{-\beta}} \quad (12)$$

Dubkov A.A., Malakhov A.N., Saichev A.I.
Radiophys. Quant. Electr. 2000. V.43, 335

For sufficiently high barriers $\beta \gg 1$ we have

$$\tau_c \sim e^{\Delta U/D}$$

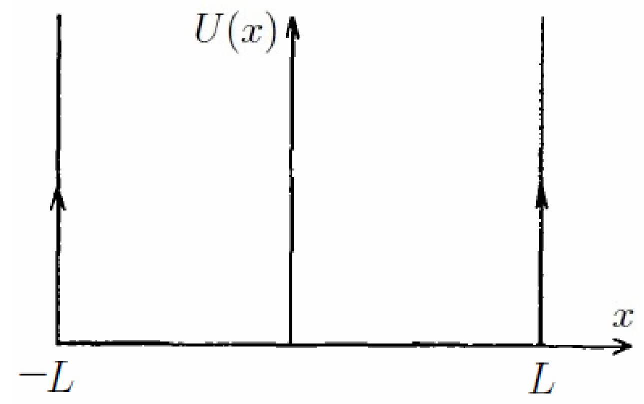
Arrhenius-Kramers law!



The spectral and correlation characteristics of Brownian diffusion

Result for an infinitely deep rectangular potential well

$$U(x) = \begin{cases} 0, & |x| < L, \\ \infty, & |x| > L \end{cases}$$



$$\tau_K = \frac{2L^2}{5D} \qquad \textbf{(13)}$$

The spectral power density of the coordinates of the particles

$$S(\omega) = \frac{2D}{\omega^2}\left(1 - \frac{1}{L}\sqrt{\frac{D}{2\omega}} \cdot \frac{\sinh L\sqrt{2\omega/D} + \sin L\sqrt{2\omega/D}}{\cosh L\sqrt{2\omega/D} + \cos L\sqrt{2\omega/D}}\right) \qquad \textbf{(14)}$$

A.A. Dubkov, V.N. Ganin, B. Spagnolo, Acta Phys.Pol. B 35, 1447 (2004)

For piece-wise linear potentials the correlation function can not be expanded in Taylor power series !!!

$$K'[0] = \langle x\hat{L}^{+}x\rangle_{st} = \langle x\,U'(x)\rangle_{st} = -D \qquad \textbf{(15)}$$

Anomalous diffusion in the form of Lévy flights

It *has* already been done

Stochastic Langevin equation for the coordinate of a particle in potential *U(x)*

$$\frac{dx}{dt} = -\frac{dU(x)}{dx} + \xi_\alpha(t) \qquad (16)$$

where: $\xi_\alpha(t)$ is the symmetric α-stable Lévy noise and α is the Lévy index ($0<\alpha<2$).

The corresponding Fokker-Planck equation for the probability density of the coordinate with fractional spatial derivative

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(\frac{dU}{dx}P\right) + D_\alpha \frac{\partial^\alpha P}{\partial |x|^\alpha} \qquad (17)$$

The analytical results for stationary probability distribution $U(x) = \gamma x^{2m}/(2m)$

$\langle x^2 \rangle < \infty$!

$$P_{st}(x) = \frac{\beta^3}{\pi\left(x^4 - x^2\beta^2 + \beta^4\right)}$$

Quartic potential (m=2)

A.V. Chechkin et al. (2002-2004)

Lévy noise can be confined by a quartic external potential!

Condition of confinement (finite variance) for the potential $U(x) = |x|^c/c$ reads

$$c > 4 - \alpha$$

Anomalous diffusion in the form of Lévy flights

It *has* already been done

A.A. Dubkov, B.Spagnolo, Acta Phys. Pol. B 38,1745 (2007)

$$P_{st}(x) = \frac{\beta^{4n+1}}{\pi(x^2+\beta^2)} \prod_{l=0}^{n-1} \frac{1}{x^4 - 2\beta^2 x^2 \cos\left[\pi(4l+1)/(4n+1)\right] + \beta^4}$$

m = 2n+1

$$P_{st}(x) = \frac{\beta^{4n-1}}{\pi} \prod_{l=0}^{n-1} \frac{1}{x^4 - 2\beta^2 x^2 \cos\left[\pi(4l+1)/(4n-1)\right] + \beta^4}$$

m = 2n

$$\beta = \sqrt[2m-1]{D/\gamma}$$

Anomalous diffusion in the form of Lévy flights

It *Has* Already Been Done

Exact results for the correlation time

- **Quartic monostable potential** $U(x) = \gamma x^4/4$

$$\tau_c = \frac{\pi}{3\sqrt{3}\sqrt[3]{\gamma D_1^2}} \qquad (18)$$

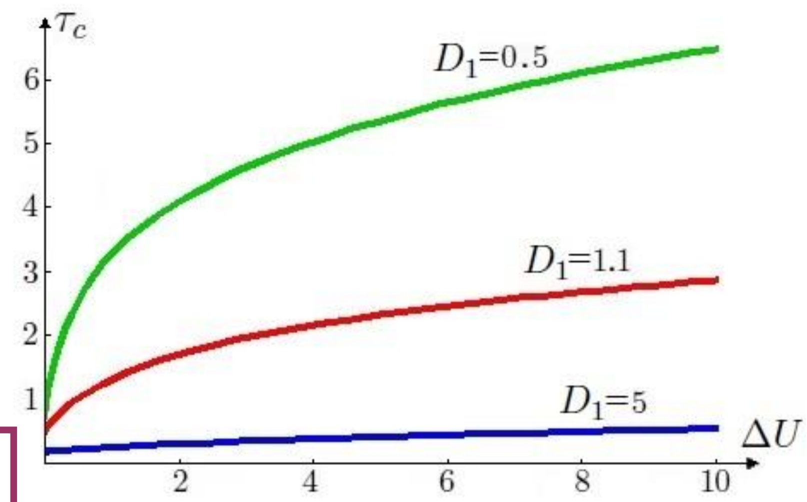
Dubkov A.A. and Spagnolo B.
Eur. Phys. J. Special Topics 2013. V.216, 31

- **Generalization for the bistable quartic potential** $U(x) = \gamma\left(x^4/4 - ax^2/2\right)\ (a > 0)$

$$\tau_c = \frac{2}{\sqrt{3}\gamma}\frac{1}{p^2 - q^2}\arctan\frac{1}{\sqrt{3}}\frac{p+q}{p-q} \qquad (19)$$

$$p = \left(D_1/(2\gamma) + \sqrt{(a/3)^3 + (D_1/(2\gamma))^2}\right)^{1/3},$$

$$q = \left(\sqrt{(a/3)^3 + (D_1/(2\gamma))^2} - D_1/(2\gamma)\right)^{1/3}, \quad p > q$$



For sufficiently high potential barriers we obtain a power dependence on the height of potential barrier unlike Brownian diffusion

$$\tau_c \approx \frac{\pi\sqrt{a}}{2D_1} \sim \sqrt[4]{\Delta U}$$

Steady-state Lévy flights in an infinitely deep rectangular potential well

The first approach: without posing any boundary conditions

The general expressions for stationary probability distribution in the case of smooth symmetrical power potential $U(x) = \frac{\gamma}{2m}\left(\frac{x}{L}\right)^{2m}$ for the anomalous diffusion in the form of Lévy flights with index Lévy α = 1 has the following form

$$P_{st}(x) = \frac{\beta^{4n+1}}{\pi(x^2+\beta^2)}\prod_{l=0}^{n-1}\frac{1}{x^4 - 2\beta^2 x^2\cos\left[\pi(4l+1)/(4n+1)\right]+\beta^4} \qquad m = 2n+1$$

$$P_{st}(x) = \frac{\beta^{4n-1}}{\pi}\prod_{l=0}^{n-1}\frac{1}{x^4 - 2\beta^2 x^2\cos\left[\pi(4l+1)/(4n-1)\right]+\beta^4} \qquad m = 2n$$

(20)

A.A. Dubkov, B.Spagnolo, Acta Phys. Pol. B 38,1745 (2007)

The smooth symmetrical power potential

$$U(x) = \frac{\gamma}{2m}\left(\frac{x}{L}\right)^{2m}$$

in the limit m→∞ →

The rectangular monostable potential

$$U(x) = \begin{cases} 0, & |x| < L, \\ \infty, & |x| > L \end{cases}$$

Steady-state Lévy flights in an infinitely deep rectangular potential well

$$P_{st}(x) = \begin{cases} \frac{1}{\pi\beta} \exp\left\{ \sum_{k=1}^{\infty} \frac{1}{2\cos\frac{\pi k}{2m-1}} \cdot \frac{1}{k} \left(\frac{x}{\beta}\right)^{2k} \right\}, & |x| < \beta; \\ \frac{1}{\pi\beta} \left(\frac{\beta}{x}\right)^{2m} \exp\left\{ \sum_{k=1}^{\infty} \frac{1}{2\cos\frac{\pi k}{2m-1}} \cdot \frac{1}{k} \left(\frac{\beta}{x}\right)^{2k} \right\}, & |x| > \beta. \end{cases} \quad (21)$$

$$\beta = L \sqrt[2m-1]{DL/\gamma}$$

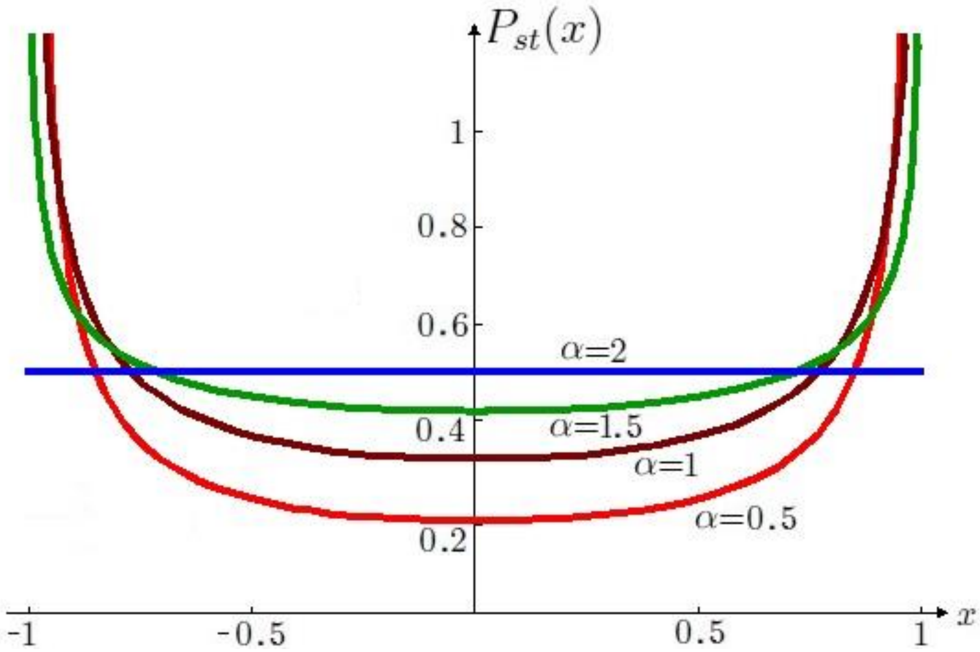
The steady-state probability distribution in Cauchy case $\alpha = 1$ has the form of well-known arcsine distribution

$$P_{st}(x) = \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{L^2 - x^2}}, & \text{если } |x| < L; \\ 0, & \text{если } |x| > L. \end{cases} \quad (22)$$

$$P_{st}(x) = \frac{(2L)^{1-\alpha}\,\Gamma(\alpha)}{\Gamma^2(\alpha/2)(L^2 - x^2)^{1-\alpha/2}} \quad (23)$$

S.I. Denisov, W. Horsthemke, P. Hänggi
Phys. Rev. E 77, 061112 (2008).

Steady-state Lévy flights in an infinitely deep rectangular potential well



Plots of stationary probability density for different values of the Lévy index . The value of the parameter *L = 1.* The case *α = 2* corresponds to usual Brownian motion*.*

Steady-state Lévy flights in an infinitely deep rectangular potential well

The second approach: with condition of impermeable boundaries

For infinitely deep rectangular potential well the boundaries at *x* = ± *L* are impermeable for particles, i.e. *P(x,t)=0* at |*x*|>*L*. In the stationary case, according to the definition of the spatial fractional derivative, the Fokker-Planck equation transforms to

$$\int_{-L}^{L} \frac{P_{st}(z) - P_{st}(x)}{|x-z|^{1+\alpha}} dz = 0 \qquad (24)$$

The solution of this integral equation has the following form for symmetric driven noise

$$P_{st}(x) = \frac{(2L)^{1-\alpha}\,\Gamma(\alpha)}{\Gamma^2(\alpha/2)(L^2 - x^2)^{1-\alpha/2}}$$

S.I. Denisov, W. Horsthemke, P. Hänggi
Phys. Rev. E 77, 061112 (2008).

Steady-state Lévy flights in an infinitely deep rectangular potential well

To find the spectral power density, the correlation function and the correlation time, first of all, we have to find function *φ(x).* Thus, substituting the operator *L+(x)* in equation

$$\hat{L}^{+}(x)\varphi(x) - p\varphi(x) = -x \qquad \textbf{(25)}$$

we need to solve the following integral equation for the function *φ(x)*

$$\int_{-L}^{L} \frac{\varphi(z) - \varphi(x)}{|x - z|^{1+\alpha}} dz - p\varphi(x) = -x. \qquad \textbf{(26)}$$

Investigation of the spectral characteristics of the steady-state Lévy flights in potential profile still remains an open problem.

The first order term in *τ* of the Taylor series expansion has an infinite coefficient in contrast to the ordinary Brownian motion (*α* = 2), where K'[0+] = −D for any potential profile

$$K'\left[0^{+}\right] = \left\langle x\hat{L}^{+}\left(x\right)x\right\rangle_{\mathrm{st}} = -\left\langle x\,U'\left(x\right)\right\rangle_{\mathrm{st}} = -\gamma\left\langle x^{2m}\right\rangle_{\mathrm{st}} = -\infty \qquad \textbf{(27)}$$

A.A. Dubkov and B. Spagnolo
Eur. Phys. J. Special Topics 216, 31–35 (2013)

Conclusions

• In the case of steady-state Lévy flights the correlation time has been found in quartic monostable and bistable potentials.

• We suggest the new method to find the steady-state characteristics of Lévy flights in rectangular potential.

UNSOLVED PROBLEMS:

• Finding the spectral characteristics of Lévy flights even in rectangular potential with Lévy index α=1.

• Asymmetric Lévy flights?

• The role of Lévy index?

THANK YOU FOR YOUR ATTENTION!