Cascade Amplification of Fluctuations

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An unsolved problem?

Anomalous diffusion

$$\langle x^2 \rangle \sim t^\alpha \ , \quad \alpha \neq 1$$

is widely observed. It is often ‘explained’ by postulating another power-law in the equation of motion (for example, a waiting-time distribution).

However, fundamental physical laws do not contain non-integer exponents. It is desirable to find more mechanisms where power-laws emerge naturally. This talk describes a new source of non-integer power-laws.

Our investigation

Separations of nearby particles in complex flows were investigated, with thermal noise:

$$\dot{x} = v(x,t) + \sqrt{2D}\eta(t)$$

$$\langle \eta(t) \rangle = 0 \qquad \langle \eta(t)\eta(t') \rangle = \delta(t - t')$$

We consider a case with negative Lyapunov exponent. Without noise, nearby trajectories coalesce. We expect that, with noise, an Ornstein-Uhlenbeck model is applicable. This implies a Gaussian distribution of separations.

$$\Delta\dot{x} = \lambda\,\Delta x + \sqrt{2D}\,\eta(t) \qquad \lambda < 0$$

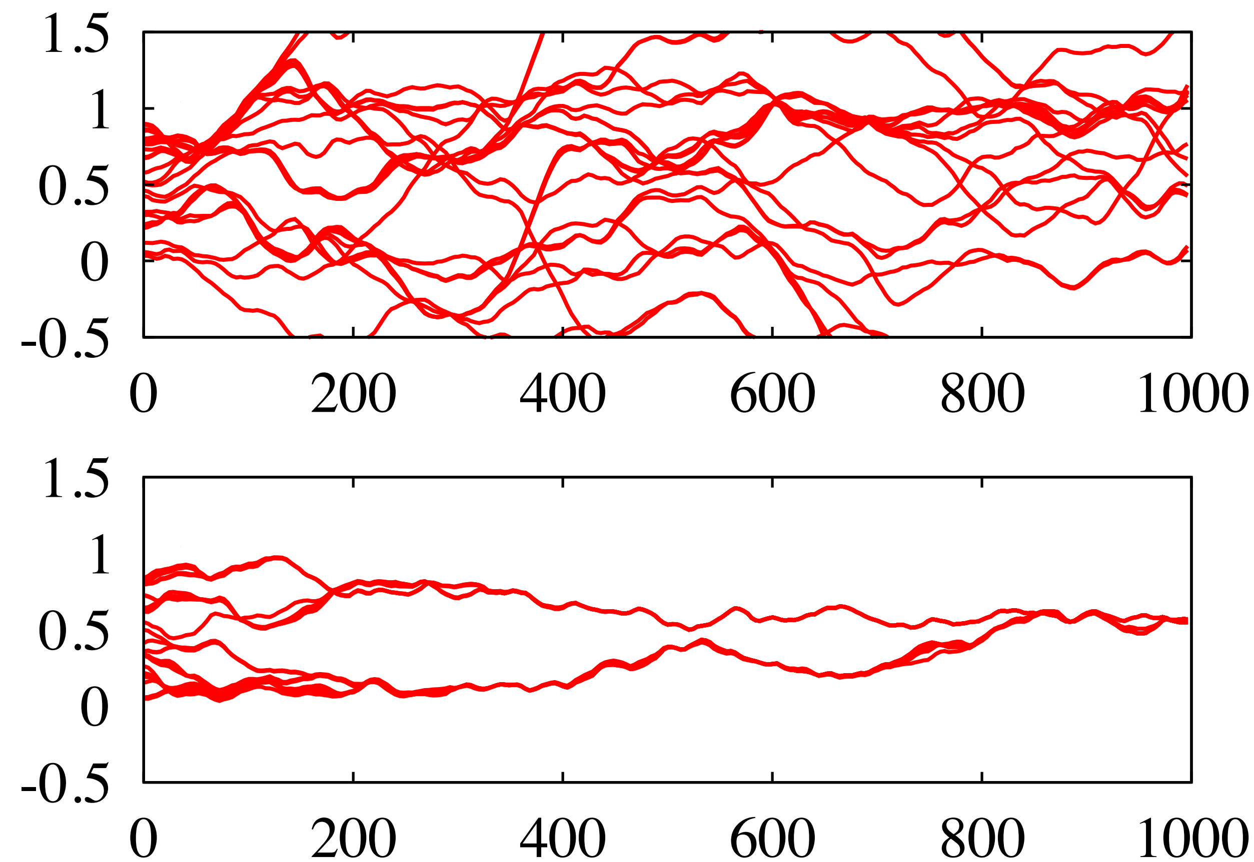
$$P_{\Delta x} = C \exp\left(-\frac{|\lambda| \Delta x^2}{4D} \right)$$

Small particles in turbulent flows

Equations of motion for small particles (1-d model):

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \gamma[u(x,t) - v] \end{aligned}$$

For sufficiently large damping the paths of particles coalesce: we are interested in this case where the Lyapunov exponent is negative:



$$Z(t) = \frac{\delta \dot{x}}{\delta x}$$

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{d}t' \ Z(t')$$

Effects of noise

Add Brownian diffusion to the model:

$$\begin{aligned} \dot{x} &= v + \sqrt{2D}\eta(t) \\ \dot{v} &= \gamma[u(x,t) - v] \end{aligned}$$

$$\langle \eta(t) \rangle = 0 \qquad \langle \eta(t)\eta(t') \rangle = \delta(t - t')$$

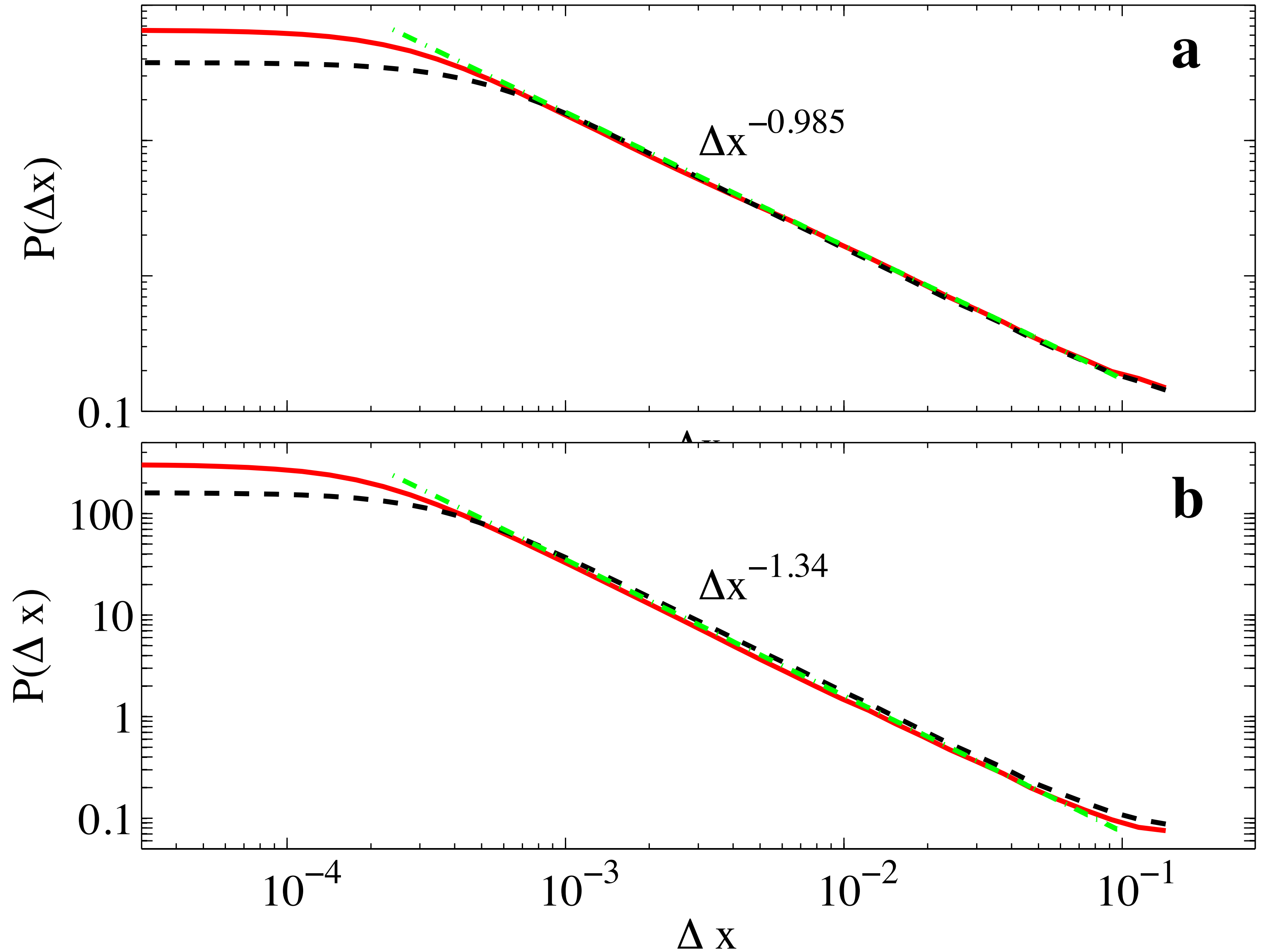
The distribution of separations of particles was found to be non-Gaussian. It has well-defined power-law tails:

$$P(\Delta x) \sim |\Delta x|^{-(1+\alpha)} \ , \quad \alpha > 0$$

Power-law distribution

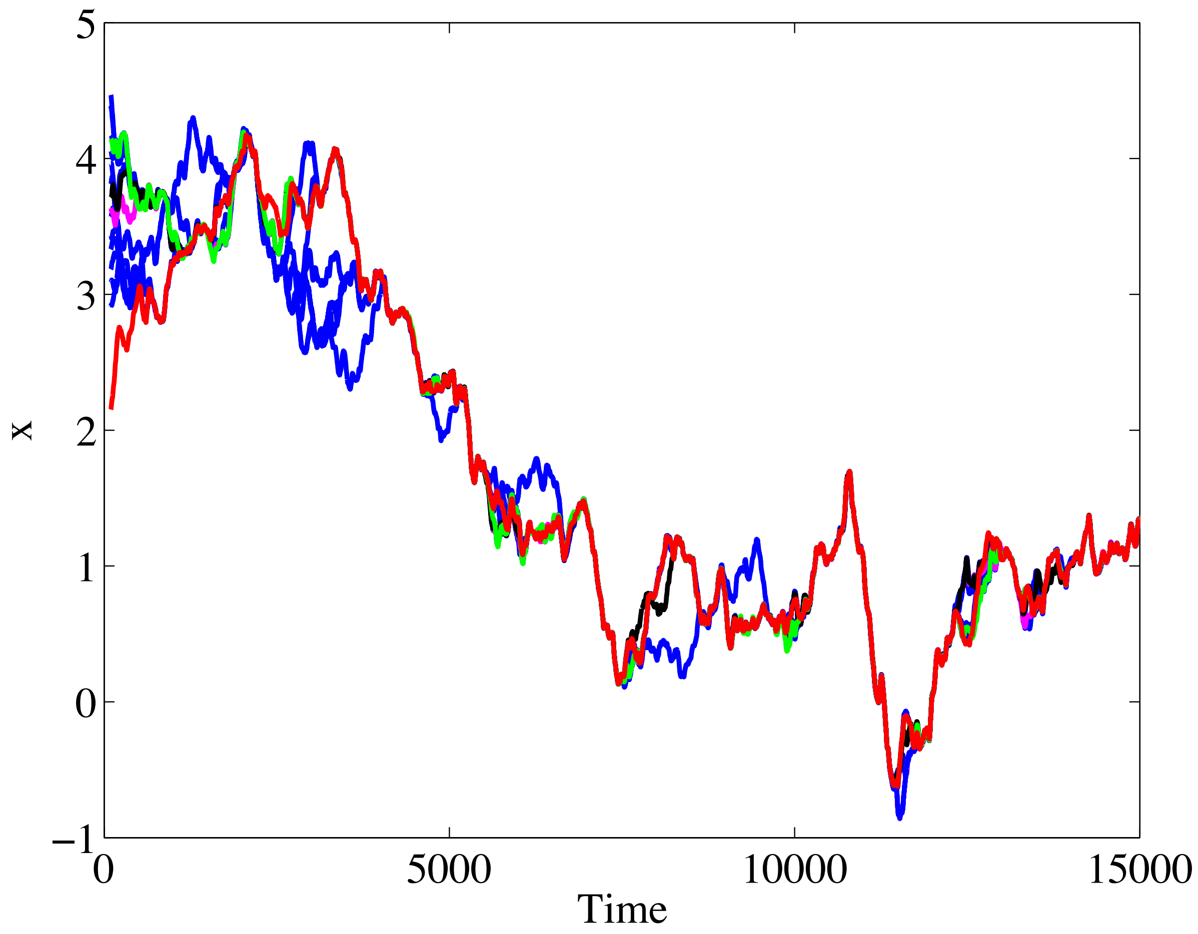
The probability density of separations is a power-law: the exponent depends upon the damping coefficient:

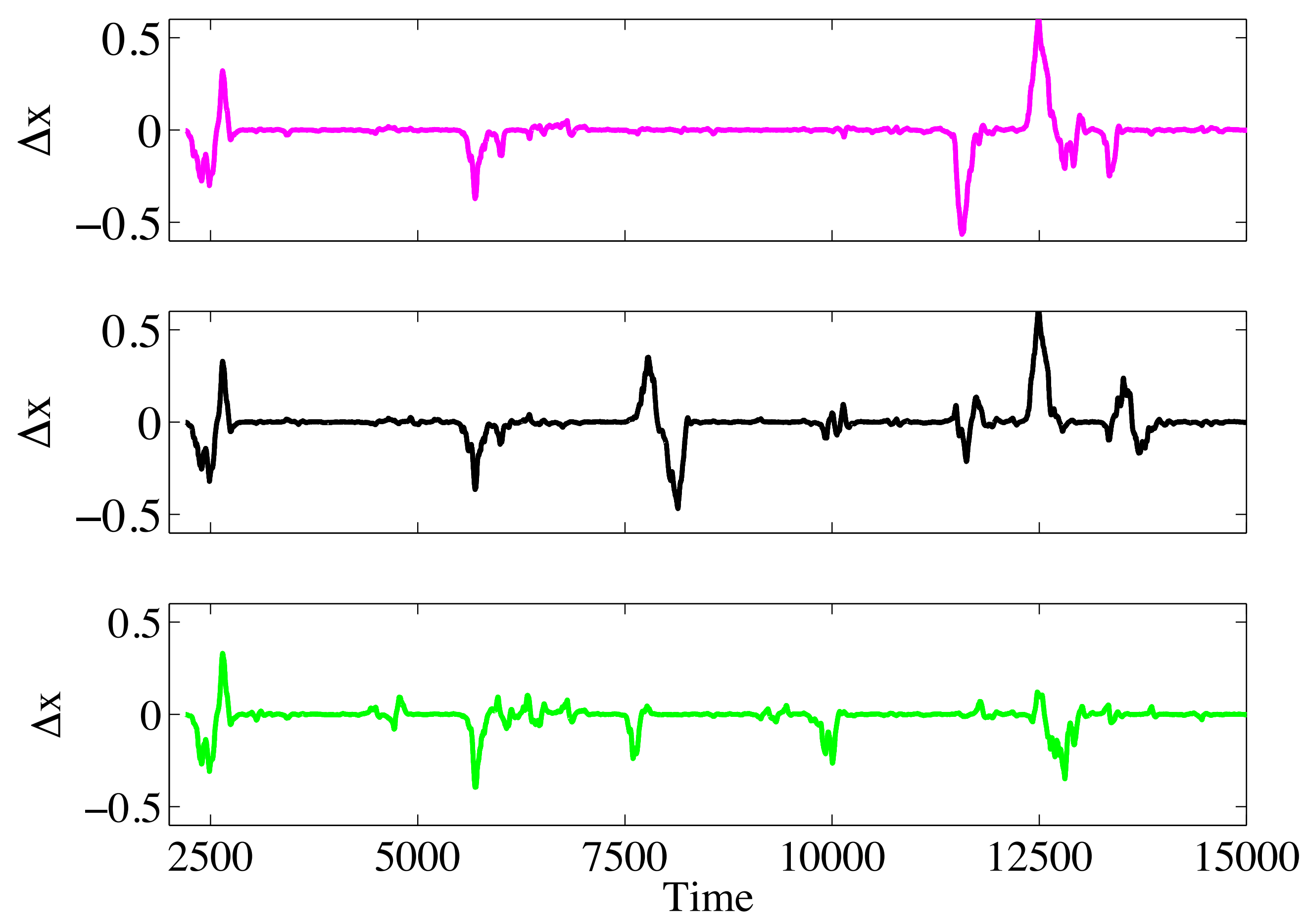
$$P_{\Delta x} \sim |\Delta x|^{-(1+\alpha)}$$



Intermittency

Numerical experiments also show that the particle separations are intermittent, with occasional large excursions.





Cascade amplification of noise

Linearised equation of motion for particle separations:

$$\delta\dot{x} = Z(t)\delta x + 2\sqrt{D}\eta(t) \qquad Z(t) = \frac{\partial v}{\partial x}(x(t), t)$$

The instantaneous Lyapunov exponent is negative most of the time, but has occasional positive excursions, with frequency independent of the particle separation. Scale invariance indicates a logarithmic variable:

$$Y = \ln(\Delta x) \qquad P_Y \sim \exp(-\alpha Y)$$

$$P_{\Delta x} \sim |\Delta x|^{-(1+\alpha)}$$

Equation for the exponent

For short correlation time, instantaneous Lyapunov exponent has equation of motion:

$$\dot{Z} = -\gamma Z - Z^2 + \sqrt{2\mathcal{D}}\,\zeta(t)$$

Seek a joint PDF in the form

$$P(Y,Z) = \exp(\alpha Y)\rho(Z)$$

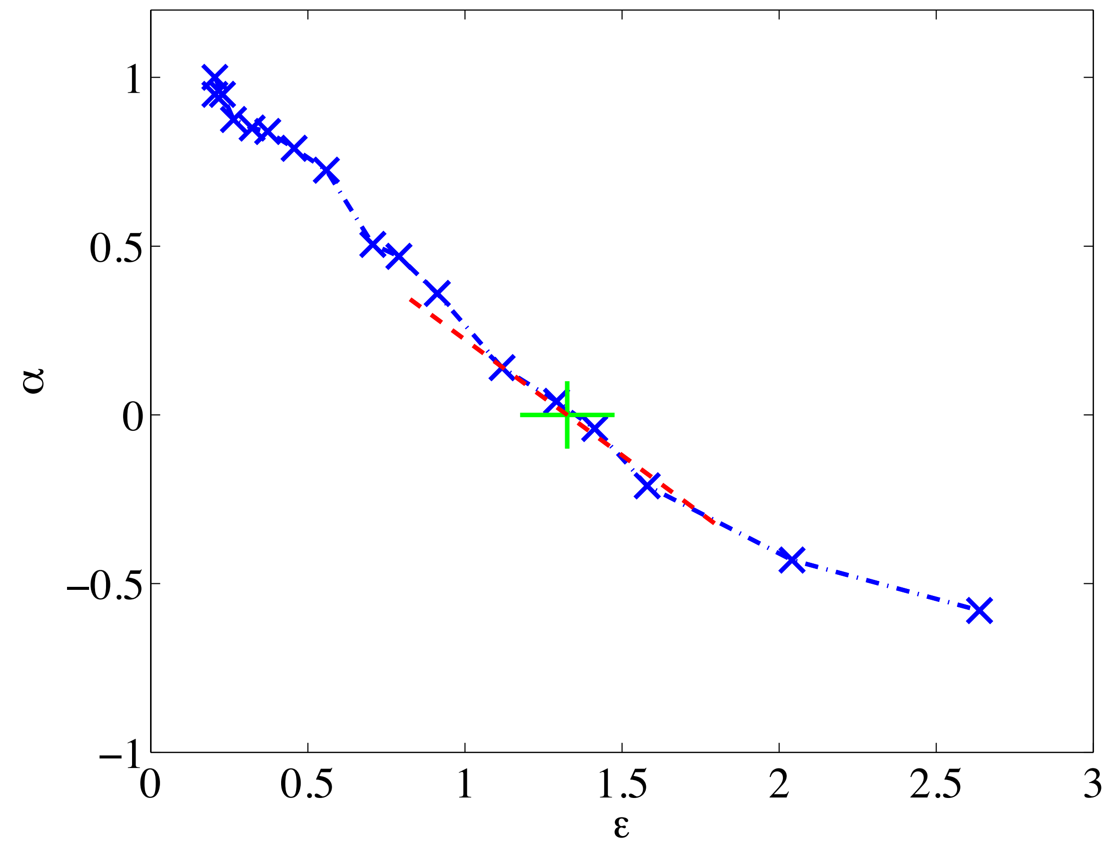
The exponent satisfies a Fokker-Planck equation and eigenvalue condition (see arXiv:1502:05855):

$$\frac{\partial}{\partial Z}\left[(\gamma Z + Z^2) + \mathcal{D}\gamma^2\frac{\partial}{\partial Z}\right]\rho(Z) + \alpha Z\rho(Z) = 0$$

$$\int_{-\infty}^{\infty} \mathrm{d}Z\ Z\,\rho(Z) = 0$$

Negative fractal dimensions

When the Lyapunov exponent is positive, the pair correlation function is a power-law, with exponent defining the correlation dimension:



$$g(\Delta x) \sim |\Delta x|^{D_2 - 1}$$

This corresponds to the expression for the particle separation due to Brownian motion if the exponent is a negative fractal dimension:

$$\alpha = -D_2$$

Summary

- The separation of particles due to Brownian fluctuations shows intermittency, and a power-law distribution.
- This is a consequence of a cascade amplification effect: there are episodes of instability which multiply the particle separation.
- The effect is very general, and it is observed in other variables (e.g. distributions of angles describing shapes of constellations of particles).
- The effect provides a new route to explain some types of anomalous diffusion, intermittency, and an interpretation of negative fractal dimensions.