$1/f$ noise arising from time-subordinated Langevin equations

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July 15, 2015

Outline

What is $1/f$ noise?

$1/f$ noise

a type of noise whose power spectral density $S(f)$ behaves like

$$S(f) \sim 1/f^{\beta}\,, \qquad \beta \text{ is close to } 1$$

- occasionally called "flicker noise"
- or "pink noise"

1/f noise

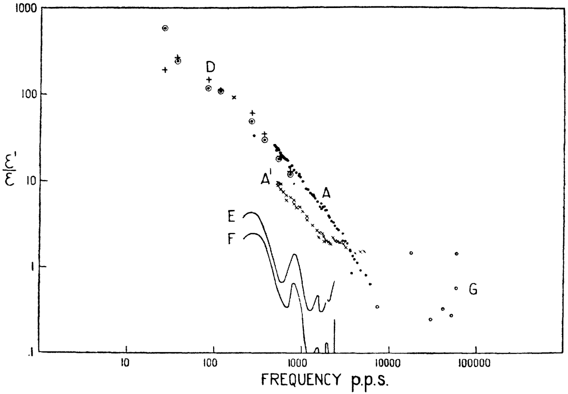


Fig. 6. Frequency variation for tube No. 2, coated filament; same data as in Fig. 4 plotted to a frequency scale; curves E and F give Hartmann's results for 2 m-a. and 20 m-a.; points G were obtained with less steady measuring circuit.

First observed (in 1925) by Johnson in vacuum tubes.

$1/f$ noise

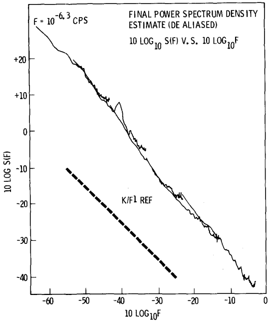


FIG. 8. Final power spectrum density estimate (dealiased).

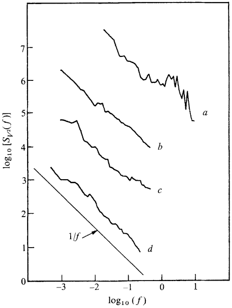
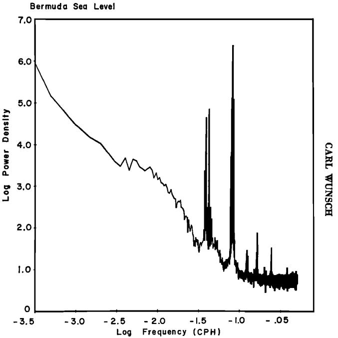


Fig. 2 Loudness fluctuation spectra, $S_{V^2}(f)$ against f for: *a*, Scott Joplin Piano Rags; *b*, classical radio station; *c*, rock station; *d*, news and talk station.



Fluctuations of signals exhibiting $1/f$ behavior of the power spectral density at low frequencies have been observed in a wide variety of physical, geophysical, biological, financial, traffic, Internet, astrophysical and other systems.

$1/f$ noise

Many mathematical models:

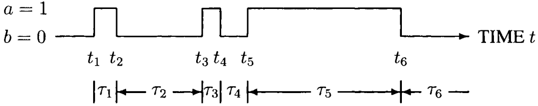
- Superposition of relaxation processes

$$S(f) = \int_{\gamma_1}^{\gamma_2} \frac{N}{\gamma^2 + \omega^2} \mathrm{d}\gamma \approx \frac{\pi N}{2\omega}, \qquad \gamma_1 \ll \omega \ll \gamma_2$$

- Dynamical systems at the edge of chaos

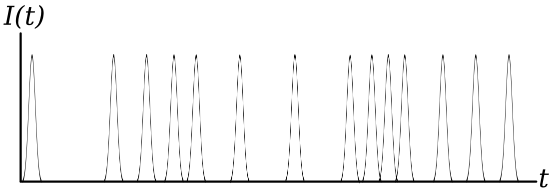
$$x_{n+1} = x_n + x_n^z \mod 1$$

- Alternating fractal renewal process



- Self-Organized Criticallity

Particular model of $1/f$ noise: point process



- The signal of the model consists of pulses or events

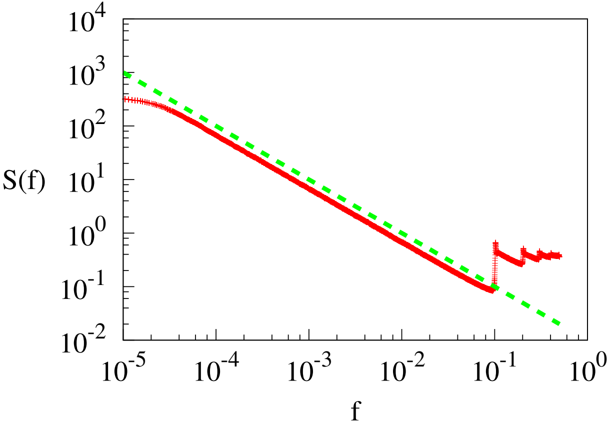
$$I(t) = a \sum_k \delta(t - t_k)$$

- Point processes arise in different fields such as physics, economics, ecology, neurology, seismology, traffic flow, financial systems and the Internet.

Correlated inter-pulse durations

Inter-pulse durations perform a random walk:

$$\tau_{k+1} = \tau_k \pm \sigma$$



Correlated inter-pulse durations

The spectrum is

$$S(f) = \frac{\nu}{f} P_{\tau}(\tau_{\mathrm{min}})$$

in the frequency range

$$\frac{\sigma^2}{\tau_{\mathrm{max}}^3} \ll f \ll \min\left(\frac{\sigma^2}{\tau_{\mathrm{min}}\tau_{\mathrm{max}}^2}, \frac{1}{\tau_{\mathrm{max}}}\right)$$

where $P_{\tau}(\tau)$ is the PDF of inter-pulse durations and

$$\sigma^2 = \int P(\tau_k|\tau_{k-1})(\tau_k - \tau_{k-1})^2 \,\mathrm{d}\tau_k$$

Point processes

- More general equation

$$\tau_{k+1} = \tau_k + \gamma \tau_k^{2\mu-1} + \sigma \tau_k^{\mu} \varepsilon_k$$

- Allows to obtain power-law exponent β in the spectrum different from 1.
- Used for modeling of the internote interval sequences of the musical rhythms

  D. J. Levitin, P. Chordia, and V. Menon, Proc. Natl. Acad. Sci. U.S.A. **109**, 3716 (2012).

Conclusion

One of possible origins of $1/f$ noise

Brownian motion in time axis leads to $1/f$ noise

Question

Can this way to $1/f$ noise be applied not only to a sequence of pulses?

The main idea

In a sequence of pulses the pulse number can be interpreted as an internal time.

- Start from a stochastic differential equation
- Interpret the time as an internal parameter.
- Add an additional equation relating the physical time to the internal time.
- Increments of the physical time should be a power-law function of the magnitude of the signal.

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In a sequence of pulses the pulse number can be interpreted as an internal time.

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Why two times?

- Impurities and regular structures in a medium results in a transport of variable speed, the particle may be trapped for some time or accelerated.
- The waiting time can depend on the particle position
- or on the intensity of the signal.
- Example: a diffusion on fractals and multifractals.

Time-subordinated Langevin equations

We consider the situation when the increments of the physical time are deterministic

$$\begin{aligned} \mathrm{d}x_\tau &= F(x_\tau)\mathrm{d}\tau + \mathrm{d}W_\tau \\ \mathrm{d}t_\tau &= g(x_\tau)\mathrm{d}\tau \end{aligned}$$

One can reduce the system of equations to a single equation in physical time with a multiplicative noise

$$\mathrm{d}x_t = \frac{F(x_t)}{g(x_t)}\mathrm{d}t + \frac{1}{\sqrt{g(x_t)}}\mathrm{d}W_t$$

Only positive values of x

We choose the function $g(x)$ as a power-law function of x:

$$\mathrm{d}t_\tau = x^{-2\eta}\mathrm{d}\tau$$

A simple Brownian motion

$$\mathrm{d}x_\tau = \mathrm{d}W_\tau$$

restricted to a interval between x_{min} and x_{max} leads to the equation in the physical time

$$\mathrm{d}x_t = x_t^\eta \mathrm{d}W_t$$

Bessel process

A Bessel process

$$\mathrm{d}x_\tau = \left(\eta - \frac{\lambda}{2}\right)\frac{1}{x_\tau}\mathrm{d}\tau + \mathrm{d}W_\tau$$

leads to the equation in the physical time

$$\mathrm{d}x_t = \left(\eta - \frac{\lambda}{2}\right)x_t^{2\eta-1}\mathrm{d}t + x_t^{\eta}\mathrm{d}W_t$$

Geometric Brownian motion

Geometric Brownian motion

$$\mathrm{d}x_\tau = \left(\eta - \frac{\lambda}{2}\right) x_\tau \mathrm{d}\tau + x_\tau \mathrm{d}W_\tau$$

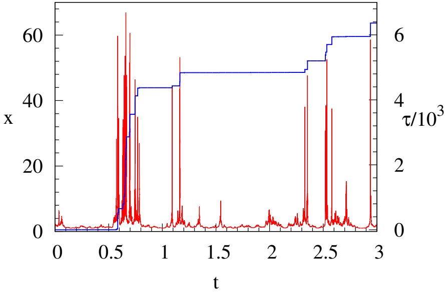
together with the relation between the internal time and the physical time

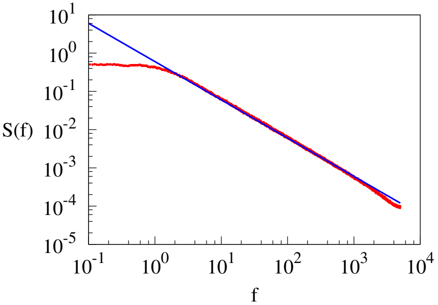
$$\mathrm{d}t_\tau = x^{-2(\eta-1)}\mathrm{d}\tau$$

leads to the same equation in the physical time

$$\mathrm{d}x_t = \left(\eta - \frac{\lambda}{2}\right) x_t^{2\eta-1} \mathrm{d}t + x_t^{\eta} \mathrm{d}W_t$$

Numerical example





Generated signal (red line) together with the corresponding internal time (blue line). The parameters are $\eta = 5/2$ and $\lambda = 3$

Spectrum of the signal (red curve). Blue line shows the slope $1/f$

Nonlinear SDEs

$$\mathrm{d}x_t = \left(\eta - \frac{\lambda}{2}\right) x_t^{2\eta-1}\mathrm{d}t + x_t^{\eta}\mathrm{d}W_t$$

- This nonlinear SDE has been proposed in

  B. Kaulakys and J. Ruseckas, Phys. Rev. E **70**, 020101(R) (2004).

  B. Kaulakys and J. Ruseckas, V. Gontis, and M. Alaburda, Physica A **365**, 217 (2006).

- Such nonlinear SDEs have been used to describe signals in socio-economical systems

  V. Gontis, J. Ruseckas and A. Kononovicius, Physica A **389**, 100 (2010).

  J. Mathiesen, L. Angheluta, P. T. H. Ahlgren and M. H. Jensen, Proc. Natl. Acad. Sci. **110**, 17259 (2013).

Estimation of spectrum from scaling properties

$$\mathrm{d}x_t = \left(\eta - \frac{\lambda}{2}\right) x_t^{2\eta-1}\mathrm{d}t + x_t^{\eta}\mathrm{d}W_t$$

- Steady state PDF has power-law form

$$P_0(x) \sim x^{-\lambda}$$

- The change of the magnitude of the stochastic variable $x \to ax$ is equivalent to the change of time scale $t \to a^{2(\eta-1)}t$.
- Trasnsition probability has a scaling property

$$P(ax', t|ax, 0) = a^{-1}P(x', a^{2(\eta-1)}t|x, 0)$$

Estimation of spectrum from scaling properties

- Autocorrelation function can be written as

$$C(t) = \int \mathrm{d}x \int \mathrm{d}x' \, xx' P_0(x) P_x(x', t|x, 0)$$

- The autocorrelation function $C(t)$ has scaling property

$$C(at) \sim a^{\beta-1} C(t)$$

with

$$\beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}$$

Both positive and negative values of x

The Ornstein-Uhlenbeck process

$$\mathrm{d}x_\tau = -\gamma x_\tau \mathrm{d}t + \mathrm{d}W_\tau$$

The relation between the internal time and the physical time

$$\mathrm{d}t = \frac{1}{(x_\tau^2 + x_0^2)^\eta}\mathrm{d}\tau$$

Resulting nonlinear SDE in physical time

$$\mathrm{d}x_t = -\gamma(x_t^2 + x_0^2)^\eta x_t \mathrm{d}t + (x_t^2 + x_0^2)^{\frac{\eta}{2}}\mathrm{d}W_t$$

Both positive and negative values of x

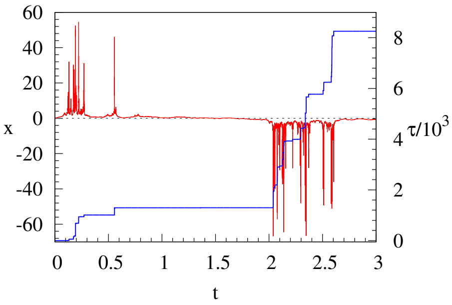
Equation

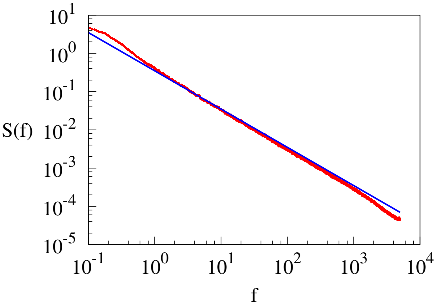
$$\mathrm{d}x_\tau = \left(\eta - \frac{\lambda}{2}\right) \frac{x_\tau}{x_\tau^2 + x_0^2} \mathrm{d}\tau + \mathrm{d}W_\tau$$

leads to SDE in the physical time

$$\mathrm{d}x_t = \left(\eta - \frac{\lambda}{2}\right) (x_t^2 + x_0^2)^{\eta-1} x_t \mathrm{d}t + (x_t^2 + x_0^2)^{\frac{\eta}{2}} \mathrm{d}W_t$$

Both positive and negative values of x





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Numerical approach

Using internal time we can obtain an effective way of solving non-linear SDEs.

For example, let us consider the non-linear SDE

$$\mathrm{d}x_t = \left(\eta - \frac{\lambda}{2}\right) x_t^{2\eta-1}\mathrm{d}t + x_t^{\eta}\mathrm{d}W_t$$

We introduce operational time τ by the equation

$$\mathrm{d}\tau_t = x_t^{2\eta}\mathrm{d}t$$

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Numerical approach

Discretizing the internal time τ with the step $\Delta\tau$ and using the Euler-Marujama approximation for the SDE we get

$$x_{k+1} = x_k + \left(\eta - \frac{\lambda}{2}\right)\frac{1}{x_k}\Delta\tau + \sqrt{\Delta\tau}\varepsilon_k\,,$$

$$t_{k+1} = t_k + \frac{\Delta\tau}{x_k^{2\eta}}$$

Summary

- $1/f$ noise can be obtained by introducing the difference between the internal time and the physical time
- and also assuming that the increments of the physical time have power-law dependence on the intensity of the signal
- This difference between physical and internal times can arise due to presence of traps or other impurities in an inhomogeneous medium
- Introduction of internal time can be an effective way to solve highly non-linear SDEs.

Thank you for your attention!