How can a player with finite memory win by switching in a sequence of Parrondo Games?

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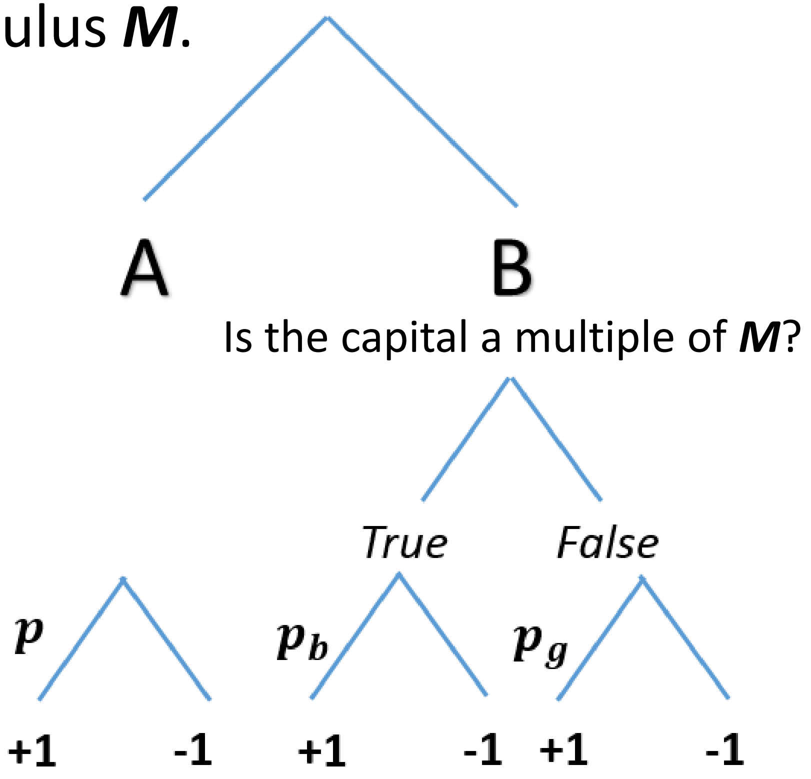
An overview of Parrondo games

- The Parrondo Game contains two games, game A and game B.
- Game A is a fair (biased) coin tossing game.
- Game B is a game depending on the capital modulus ***M***.



Random Switching and periodic switching

- For random switching, there is a probability γ to choose game A and probability $1-\gamma$ to choose game B.
  - The new game is a linear convex combination of games A and B.
  - The losing region is a non-convex set.

- For periodic switching, the player is choosing game A or B periodically.
  - For example, the ABABB… sequence.

Random Switching and periodic switching

- Generally, the game can be viewed as a finite discrete Markov Chain.
- By define

$$p(c_{t+1} \mid c_t) = \mathrm{P}(X_{t+1} = c_{t+1} \text{ Mod } 3 \mid X_t = c_t \text{ Mod } 3)$$

where c_t is player's capital at time t.

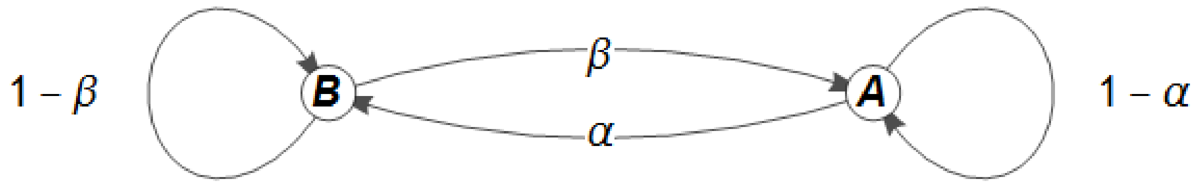
Game B transition matrix

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & p_b & 1-p_b \\ 1-p_g & 0 & p_g \\ p_g & 1-p_g & 0 \end{pmatrix} \end{array}$$

- We can find the long term behavior of the expected capital gain by finding the eigenvectors with eigenvalue 1.
- And times the payoff vector which is $(2p_b - 1, 2p_g - 1, 2p_g - 1)$

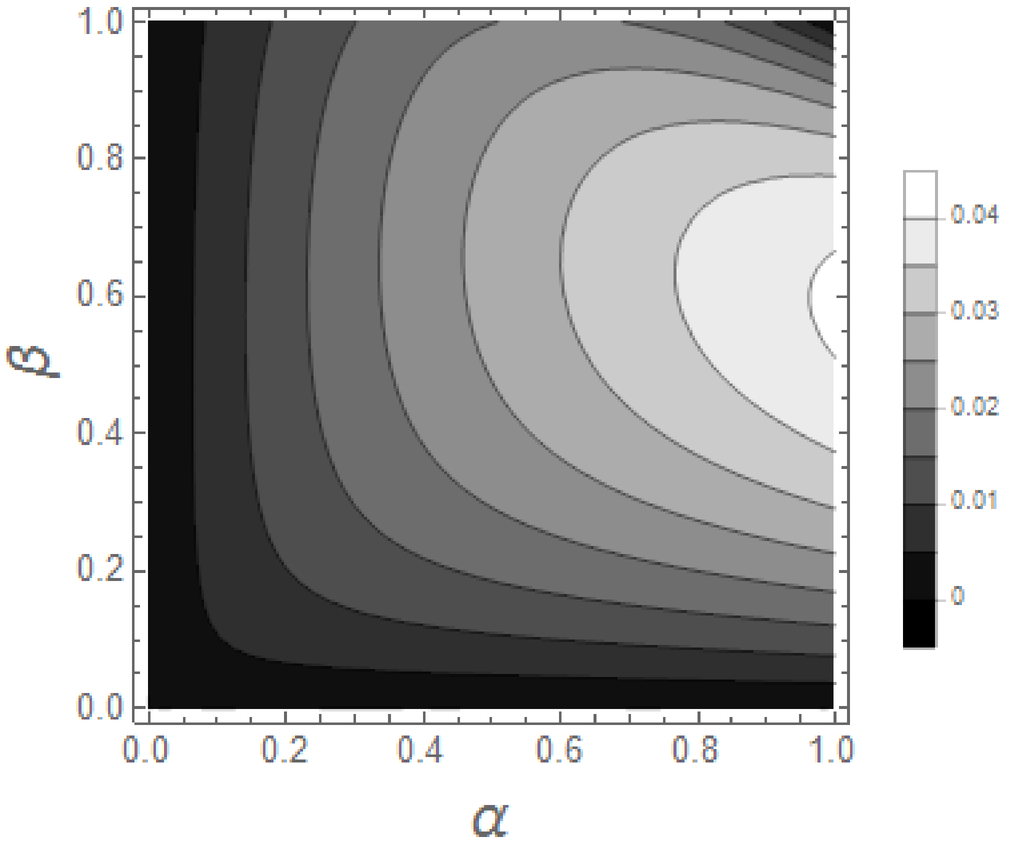
Switching with finite memory

- Consider the player has memories when playing the game,
- The simplest case is that, the player remember ***one*** game he played before.
- After each game, he/she has a probability to switch to other game conditional on the current game.



- It involves two parameters α and β instead of single γ.

Switching with finite memory for M=3



The optimal value is $\alpha = 1, \beta = 0.5907$

A contour plot for a particular set of winning probabilities

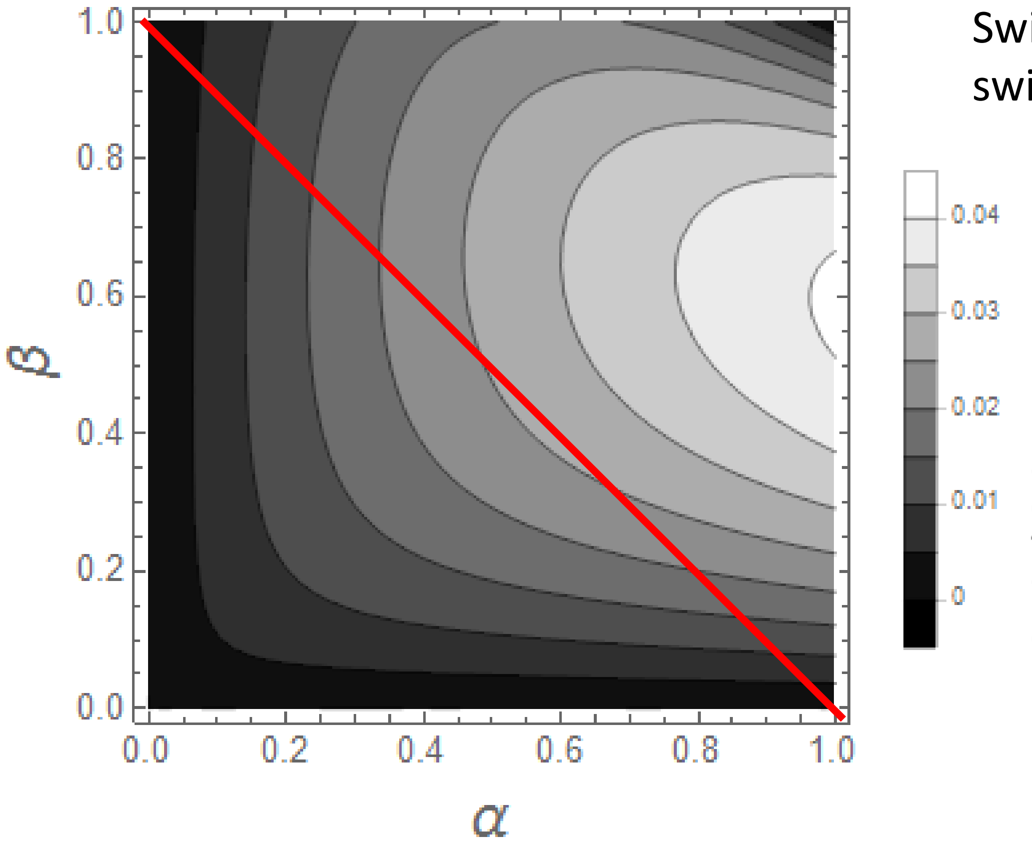
$p = 0.5 - \epsilon,$

$p_g = 0.75 - \epsilon,$

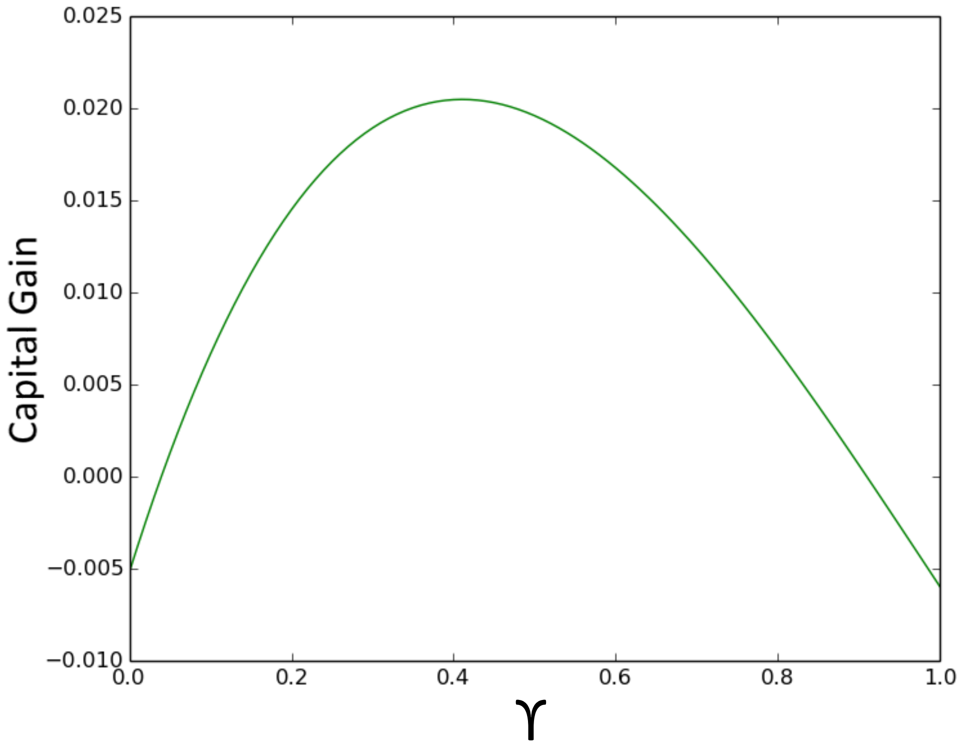
$p_b = 0.1 - \epsilon,$

$\epsilon = 0.003.$

Switching with finite memory for M=3



Switching with finite memory is reduced to random switching when $\alpha = 1 - \gamma, \beta = \gamma$.



The matrix formulation for M=3

- The game can be modelled by a 6×6 transition matrix,
- Let us denote by c_t the player's capital at time t, and g_t the player's game at time t.

$$p\left(g_{t+1}c_{t+1} \mid g_t c_t\right) = \mathrm{P}\left(X_{t+1} = c_{t+1} \text{ Mod } 3 \textit{ and } Y_{t+1} = g_{t+1} \mid X_t = c_t \text{ Mod } 3 \textit{ and } Y_t = g_t\right)$$

$$= \mathrm{P}\left(X_{t+1} = c_{t+1} \text{ Mod } 3 \mid X_t = c_t \text{ Mod } 3 \textit{ and } Y_t = g_t\right)\mathrm{P}\left(Y_{t+1} = g_{t+1} \mid \cancel{X_t = c_t \text{ Mod } 3 \textit{ and }} Y_t = g_t\right) \; *$$

$$= \mathrm{P}\left(X_{t+1} = c_{t+1} \text{ Mod } 3 \mid X_t = c_t \text{ Mod } 3 \textit{ and } Y_t = g_t\right)\mathrm{P}\left(Y_{t+1} = g_{t+1} \mid Y_t = g_t\right)$$

$$= p\left(c_{t+1} \mid c_{t,}\right)\mathrm{P}\left(Y_{t+1} = g_{t+1} \mid Y_t = g_t\right)$$

* Since the of probability of switching will not be affected by the capital state.

The matrix formulation for M=3

- Let the distribution of states be $\mathrm{U} = (A0, A1, A2, B0, B1, B2)$, we can build the matrix

$$\begin{bmatrix} 0 & p(-\alpha+1) & (\alpha-1)(p-1) & 0 & \alpha p & \alpha(-p+1) \\ (\alpha-1)(p-1) & 0 & p(-\alpha+1) & \alpha(-p+1) & 0 & \alpha p \\ p(-\alpha+1) & (\alpha-1)(p-1) & 0 & \alpha p & \alpha(-p+1) & 0 \\ 0 & \beta p_b & \beta(-p_b+1) & 0 & p_b(-\beta+1) & (\beta-1)(p_b-1) \\ \beta(-p_g+1) & 0 & \beta p_g & (\beta-1)(p_g-1) & 0 & p_g(-\beta+1) \\ \beta p_g & \beta(-p_g+1) & 0 & p_g(-\beta+1) & (\beta-1)(p_g-1) & 0 \end{bmatrix}$$

- And corresponding payoff vector is $(2p-1, 2p-1, 2p-1, 2p_b-1, 2p_g-1, 2p_g-1)$

Memories with more steps

- We can use the same approach for multiple-step-memory.
- For two-steps memory, we can write the matrix element as

$$p\left(g_{t+2}c_{t+2} \mid g_{t+1}c_{t+1}, g_t c_t\right) = p\left(c_{t+2} \mid c_{t+1}\right) \mathrm{P}\left(Y_{t+2} = g_{t+2} \mid Y_{t+1} = g_{t+1}, Y_t = g_t\right)$$

- Generally, for n-steps matrix, we need a $(3 \times 2^n)(3 \times 2^n)$ matrix
- For a matrix with dimension m, The complexity is $O(m^3)$. Not applicable for long memory sequence.

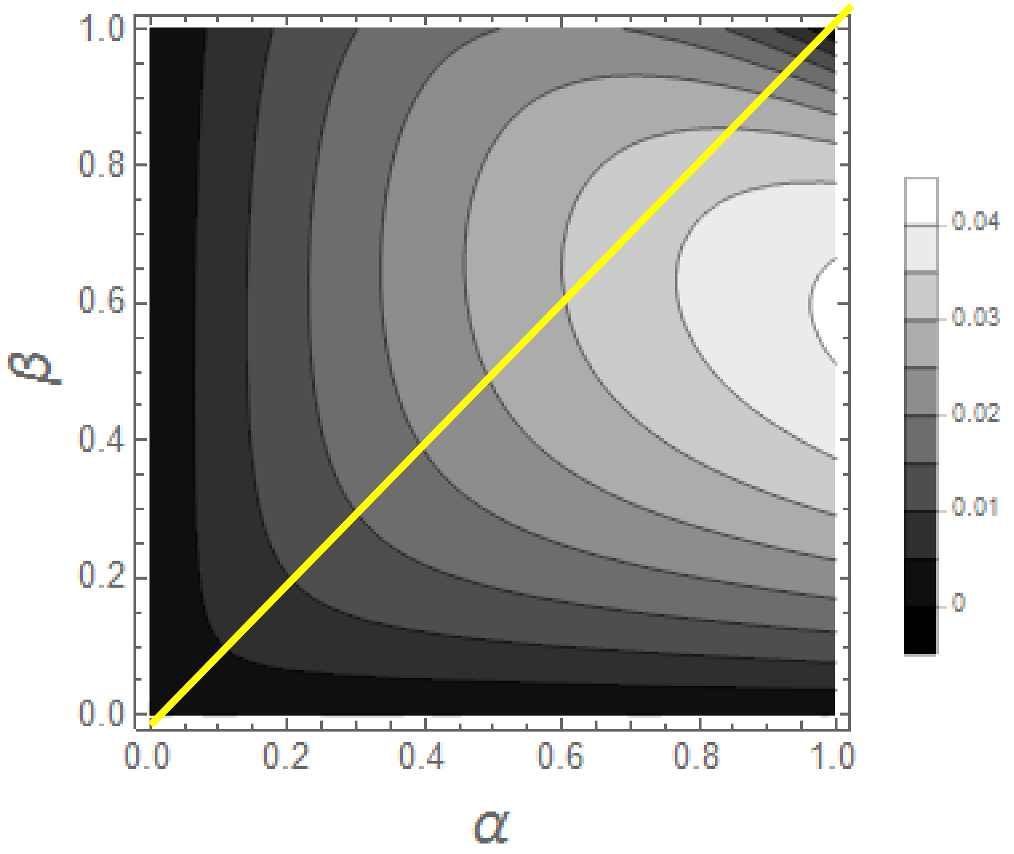
Switching with finite memory

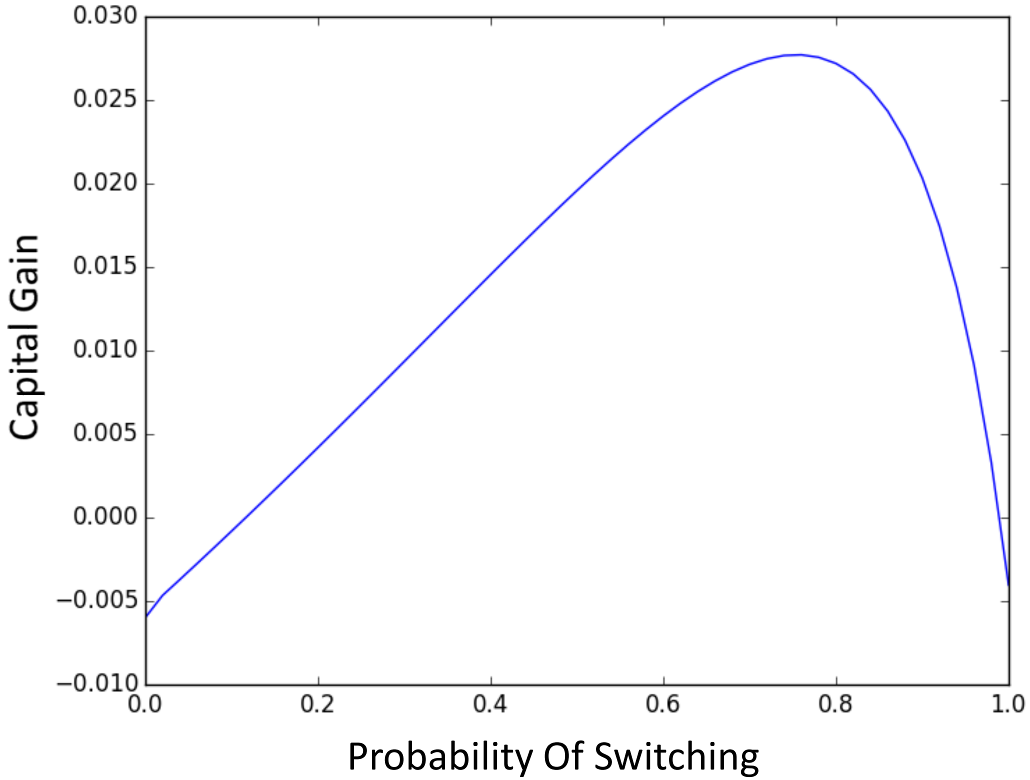
- Now consider the situation that the player cannot distinguish the game A from game B
- We may think that the player is playing the game C and game D.

- The player cannot know whether game C(D) is game A or game B.
- For one step memory, there is ***one*** switching probability.

Switching with finite memory for M=3

- It is the case when $\alpha = \beta$, the player has the constant switching probability α.

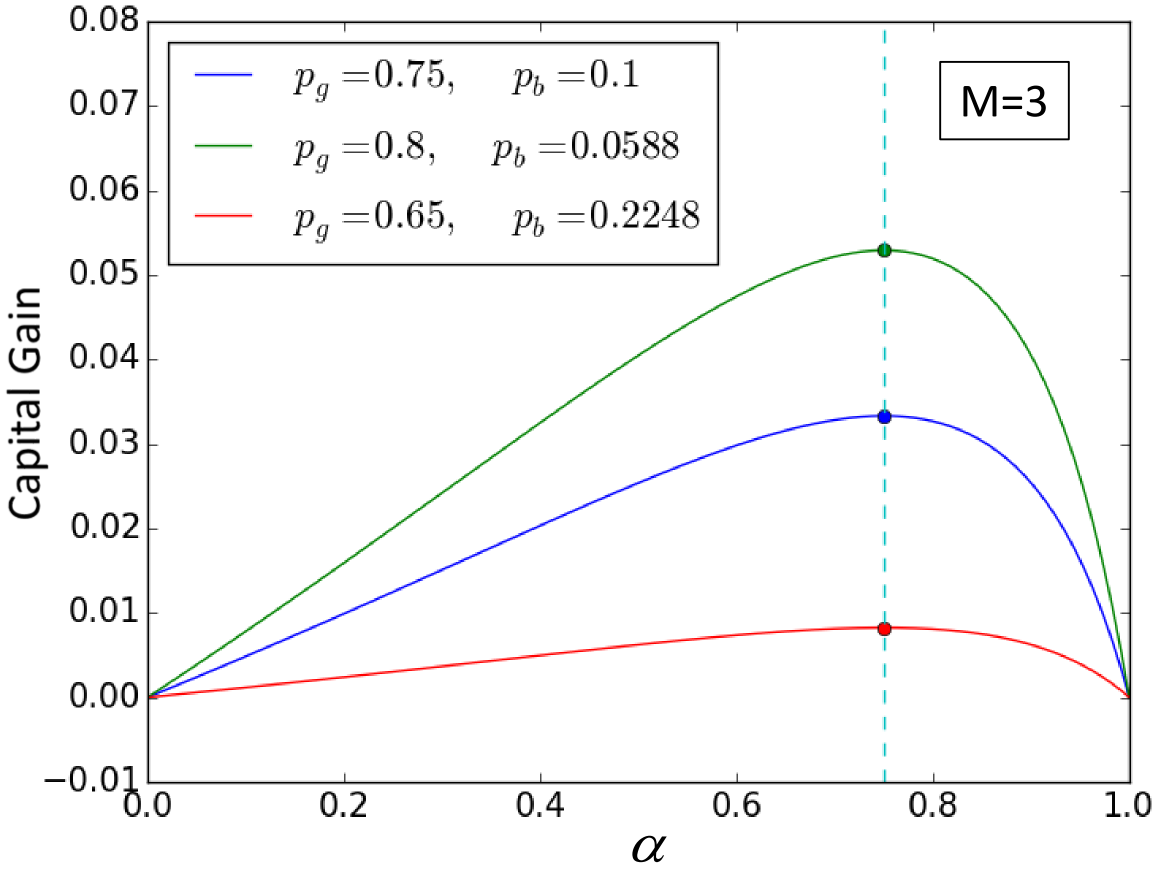




The new game again is a convex linear combination of two types of games.

Switching with finite memory

- The following are illustrations for different sets of winning probabilities (p_g, p_b)



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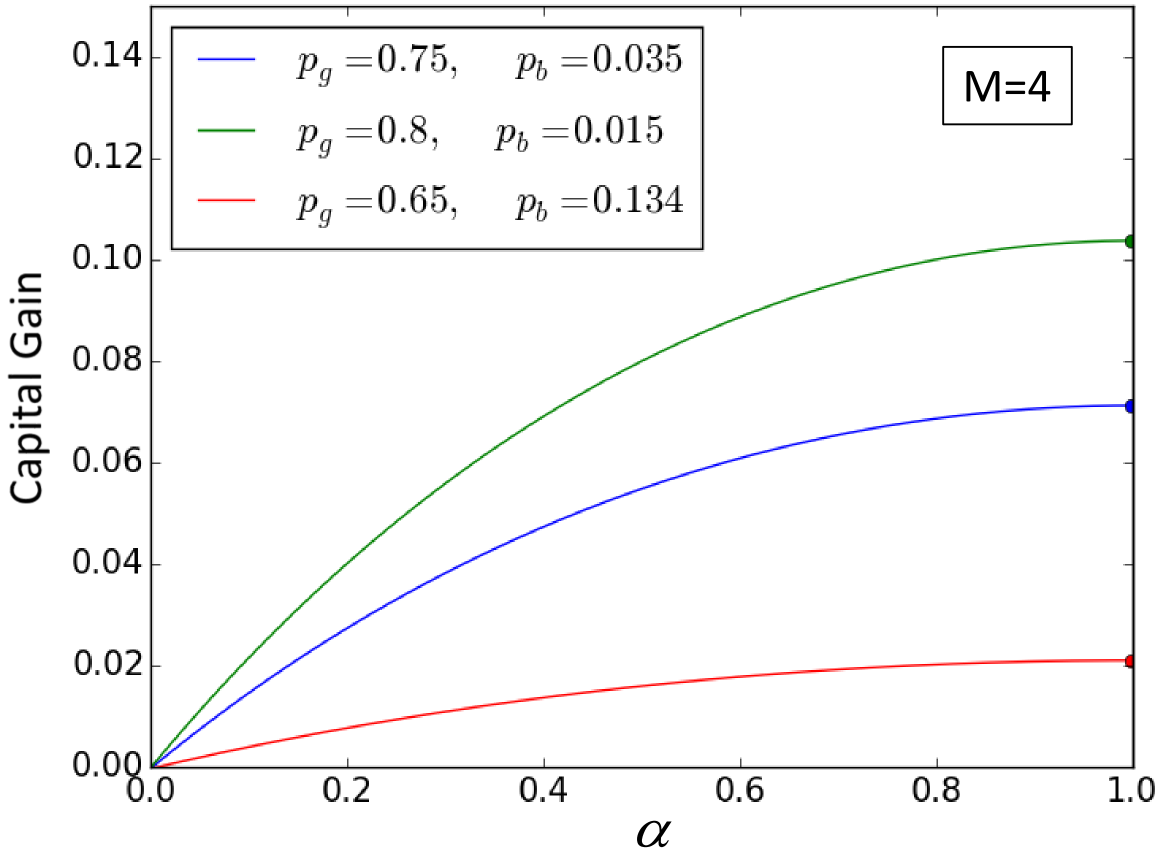

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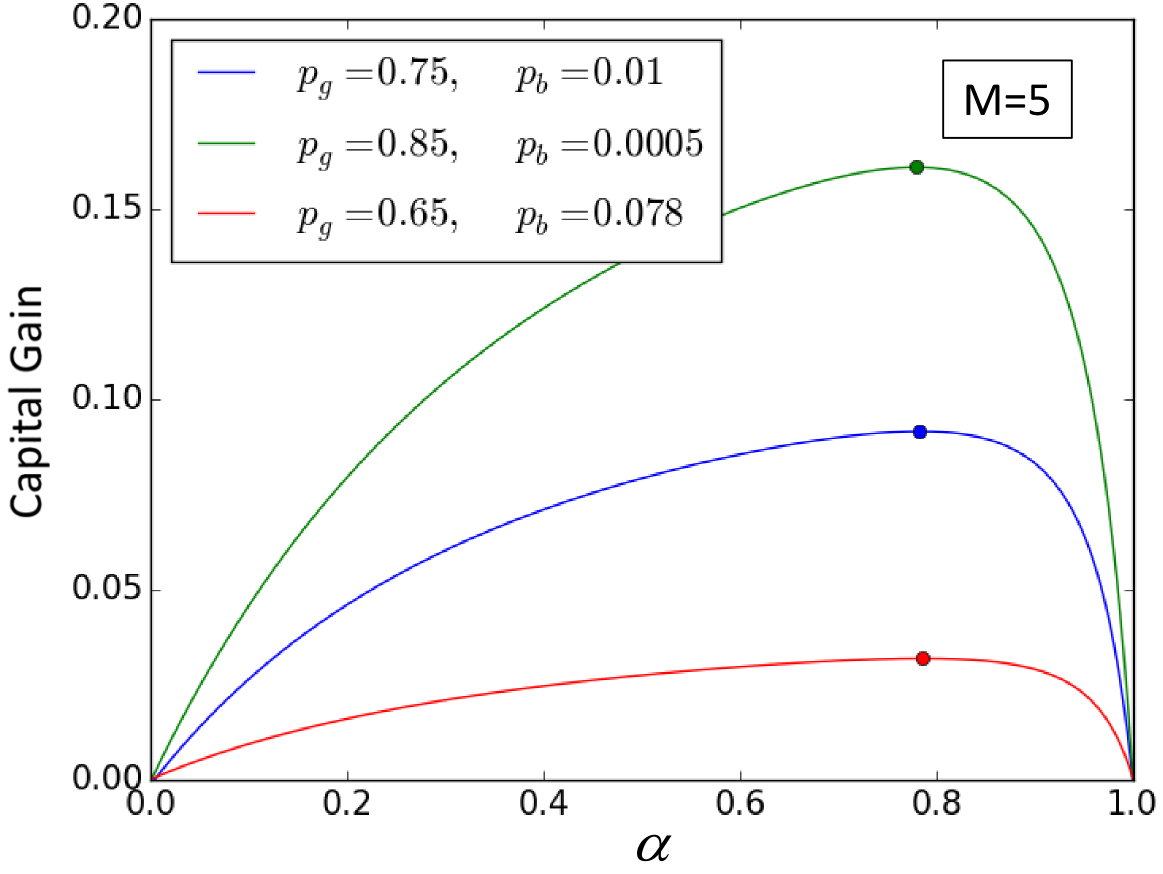

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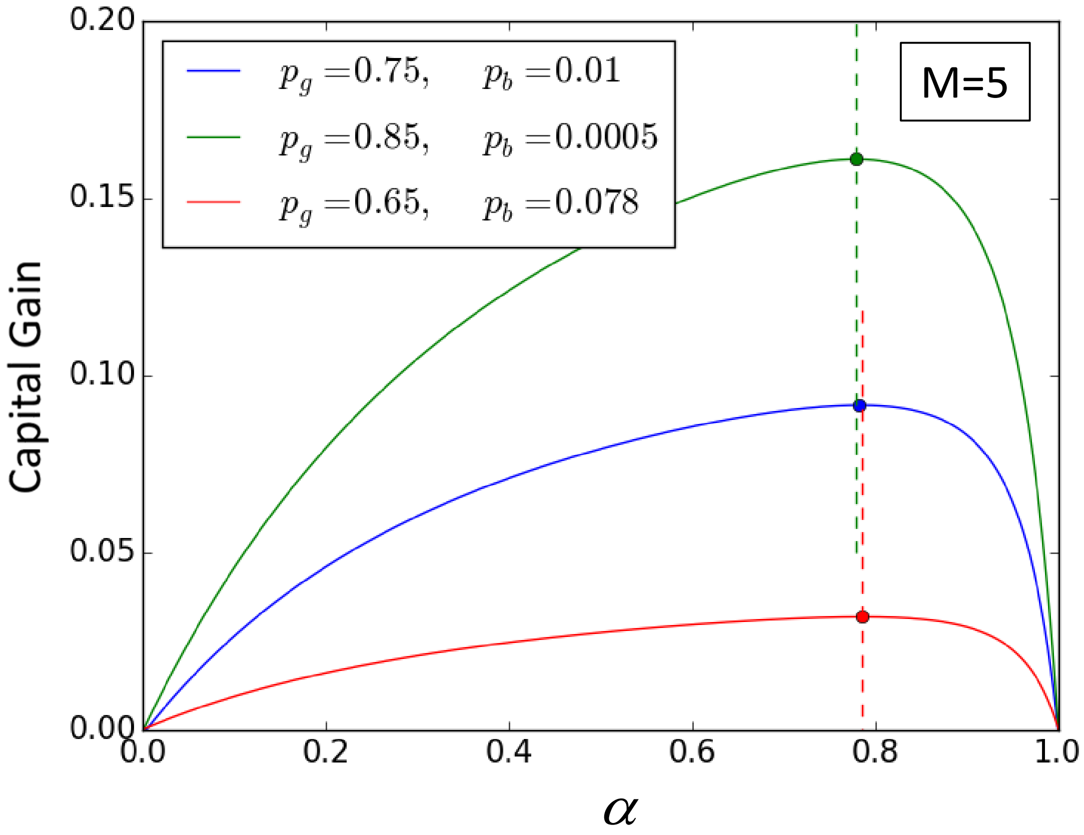
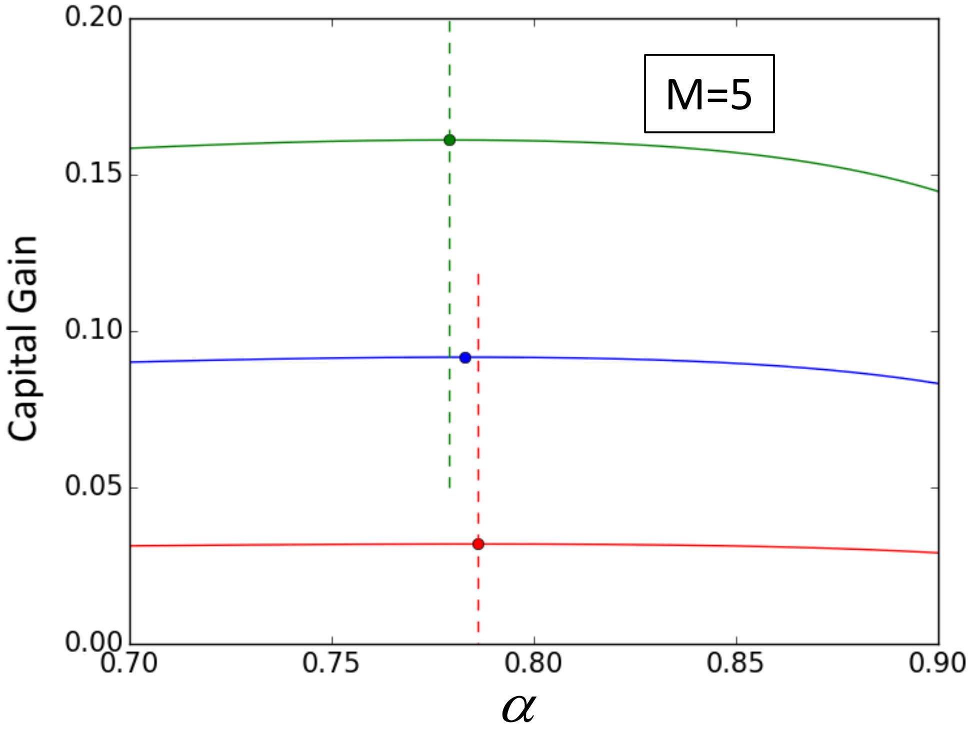
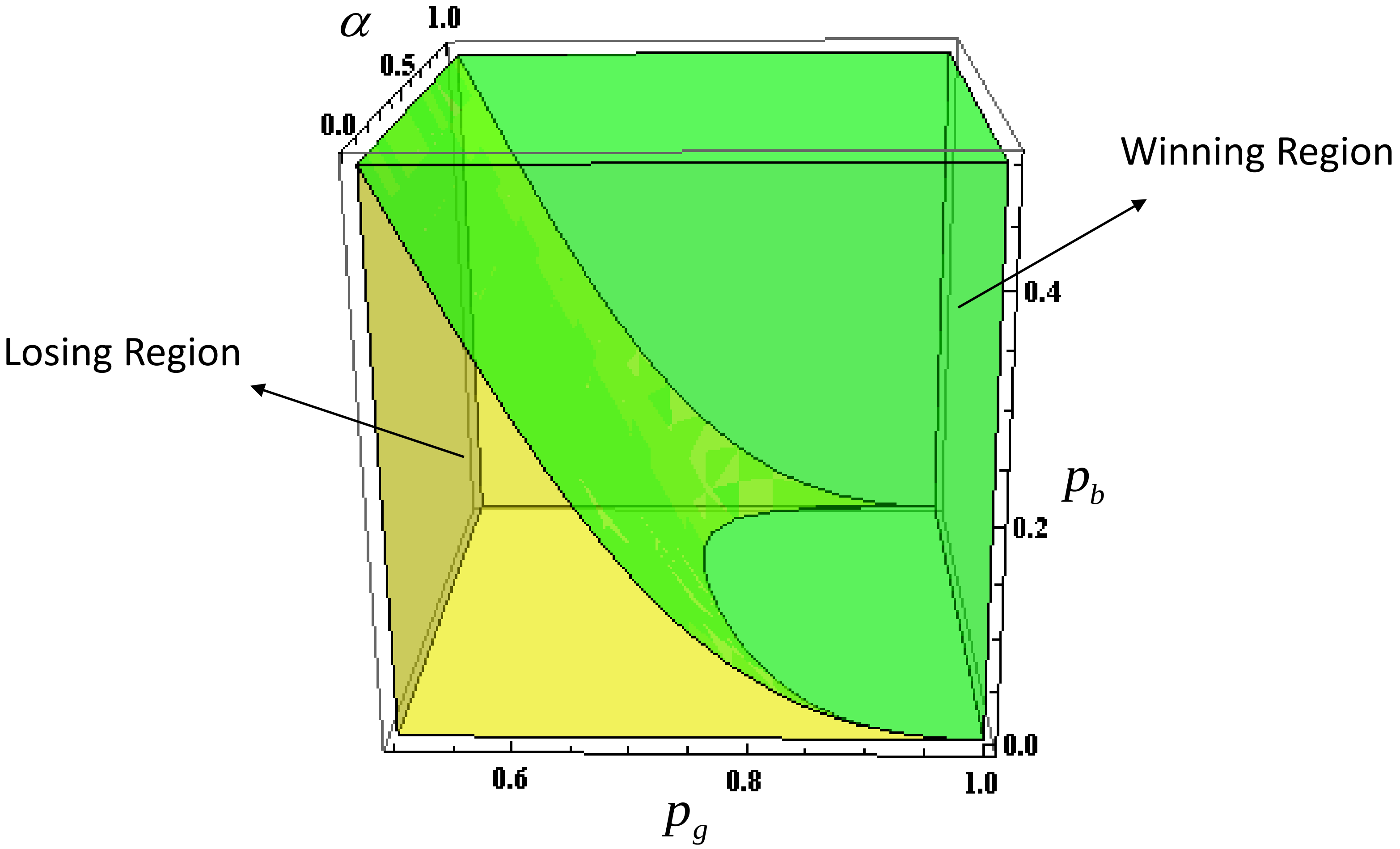




Switching with finite memory for M=3

- This type of new game again has a non-convex losing region.



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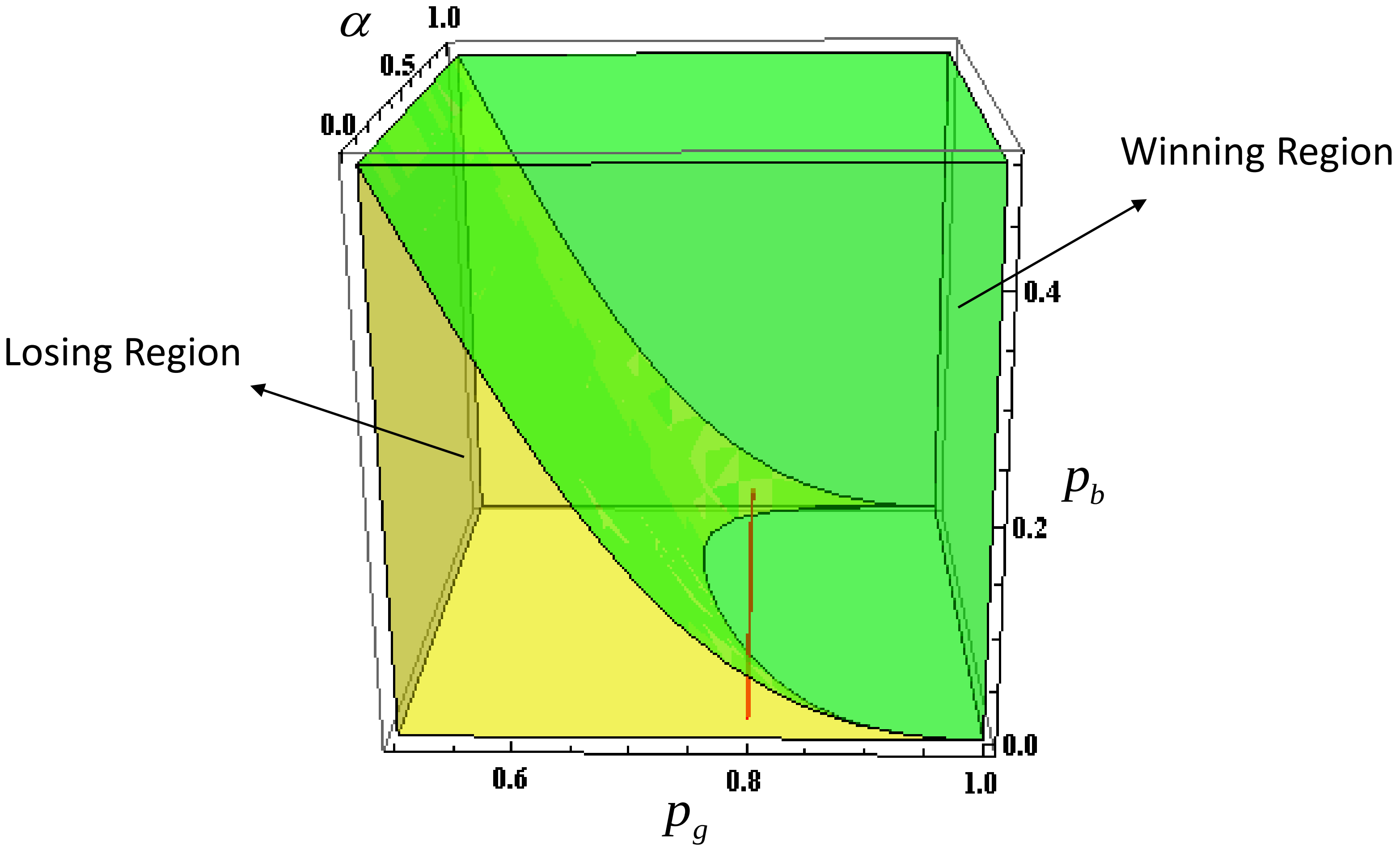

The optimizing problem

- The CD game has some interesting property.
- First, for odd M, the expected capital gain is same for $\alpha = 0$ and $\alpha = 1$ if $p = 0.5$.

The optimizing problem

- The CD game has some interesting property.
- First, for odd M, the expected capital gain is same for $\alpha = 0$ and $\alpha = 1$ if $p = 0.5$.
- Second, for M=3, We can obtain the analytical solution of optimal α

$$\alpha = \frac{\left(p^2 - p + 1\right)\left(2p_g - 1\right)}{4p^2 p_g - 2p^2 - 4pp_g + 3p + 3p_g - 2}$$

- For $p = 0.5$, the optimal α is always 3/4 for any $p_g > 0.5$ and p_b.

Summary And Future Research

Summary:

- We generalize the random switching to switching with memories.
- We focus on the case that the player with one-step memory
  - For $\alpha = \beta$, it raises a new type of Parrondo Paradox.
  - It has a unique optimal value.

Summary And Future Research

Summary:

- We generalize the random switching to switching with memories.
- We focus on the case that the player with one-step memory
  - For $\alpha = \beta$, it raises a new type of Parrondo Paradox.
  - It has a unique optimal value.

Future Research:

- Analyze different types of histories, e.g. Win/Loss sequence
- Apply the CD game concepts to group Parrondo Game.

Reference

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2. P. Flitney, D. Abbott, Phys, Rev. A ,314, 35 (2002)
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Discussion

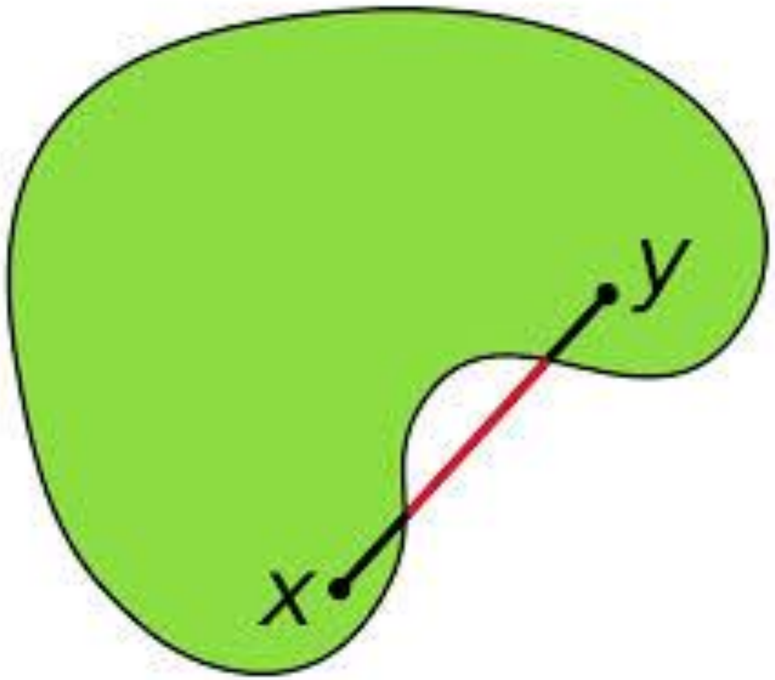
The matrix formulation

- Note that when $\alpha = 0, \beta = 0$, then transition matrix changes to

$$\begin{bmatrix} 0 & p & -p+1 & 0 & 0 & 0 \\ -p+1 & 0 & p & 0 & 0 & 0 \\ p & -p+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_b & -p_b+1 \\ 0 & 0 & 0 & -p_g+1 & 0 & p_g \\ 0 & 0 & 0 & p_g & -p_g+1 & 0 \end{bmatrix}$$

- It is reducible! We just consider the limit case $\alpha \to 0, \beta \to 0$.

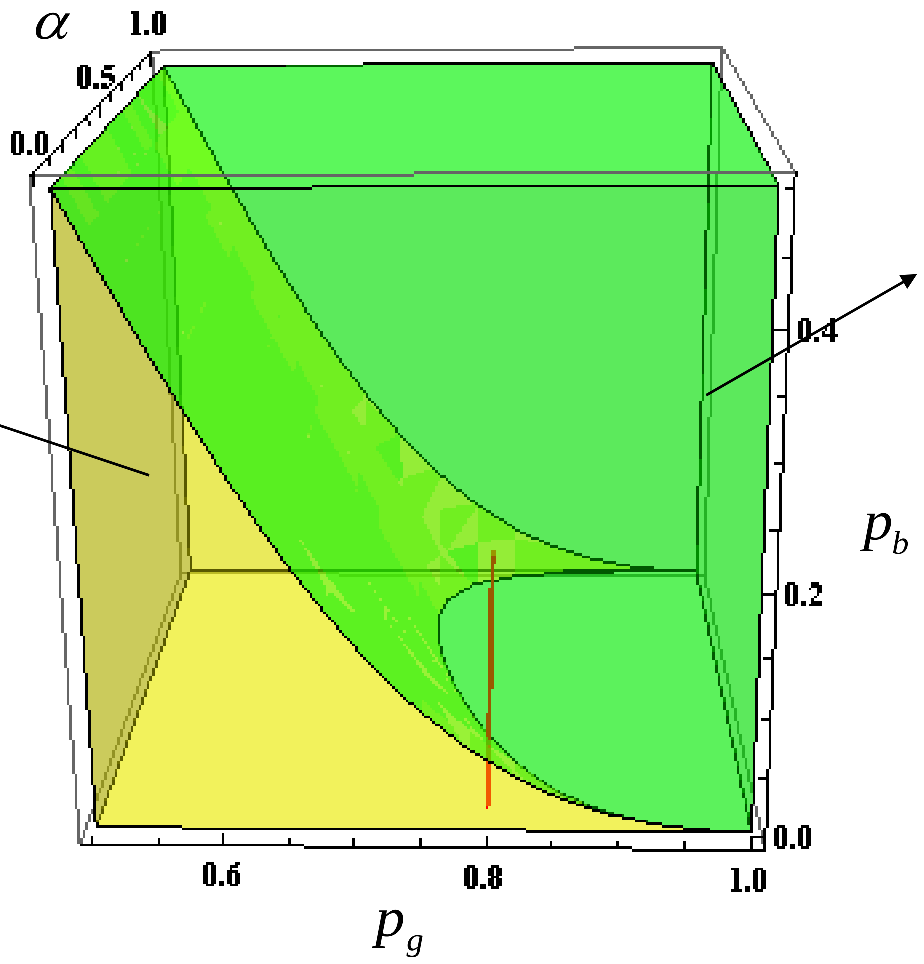
The non-convex set



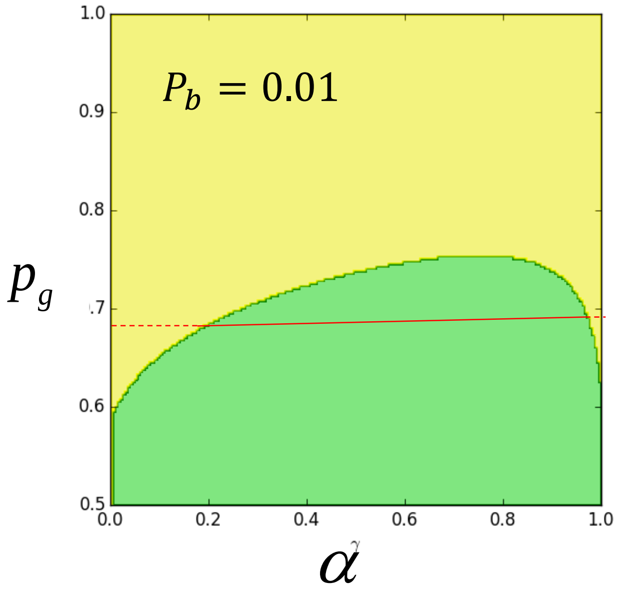
Switching with finite memory for M=3

- This type of new game again has a non-convex losing region.

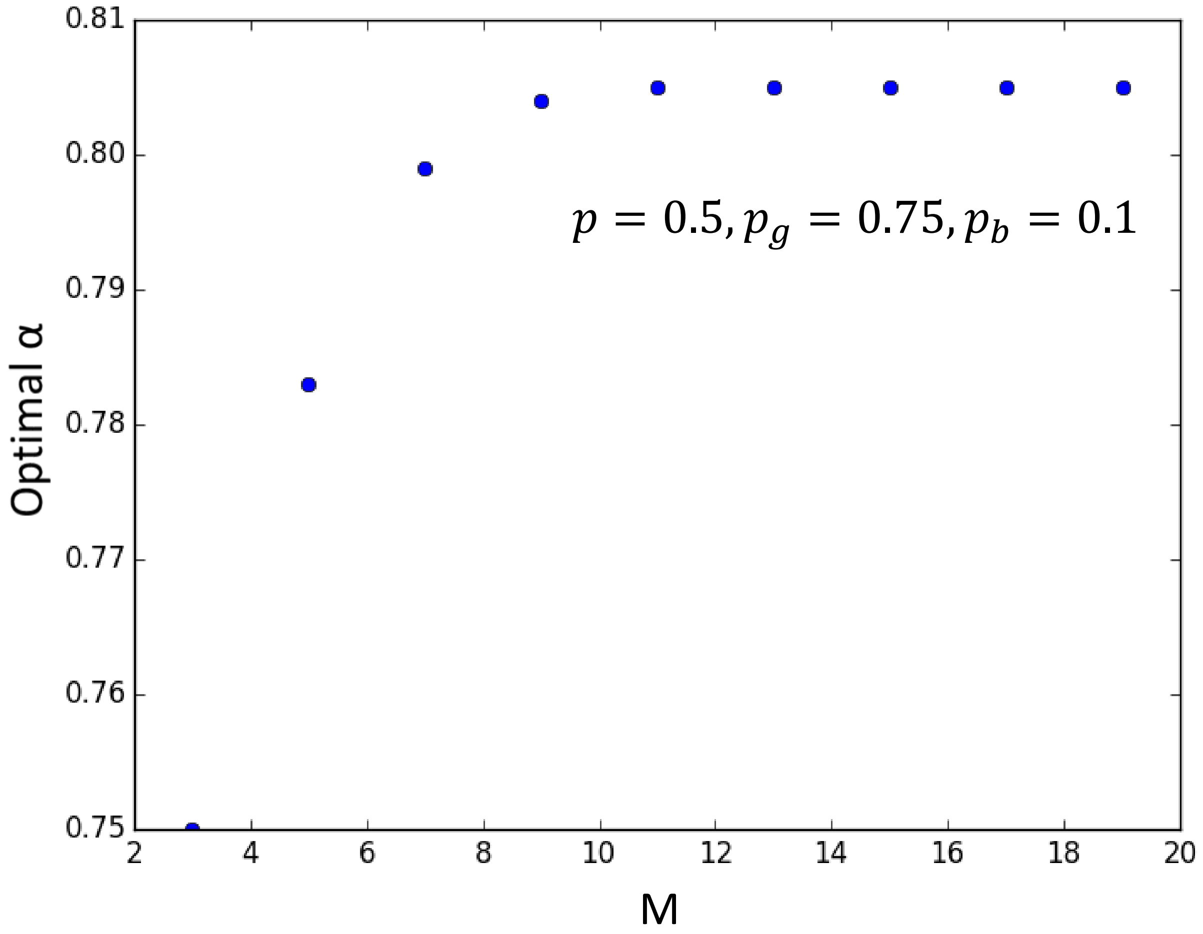
Losing Region



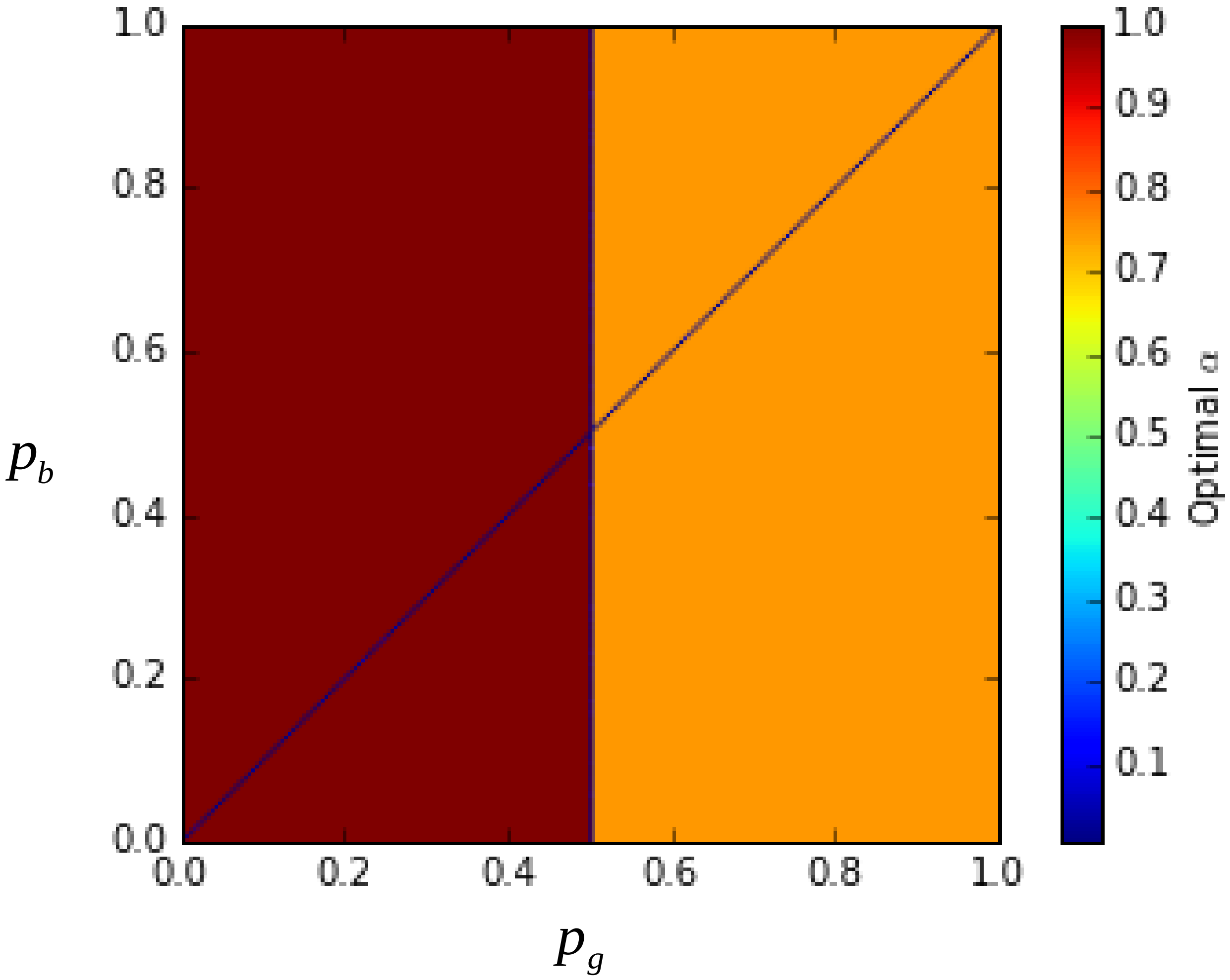
Winning Region



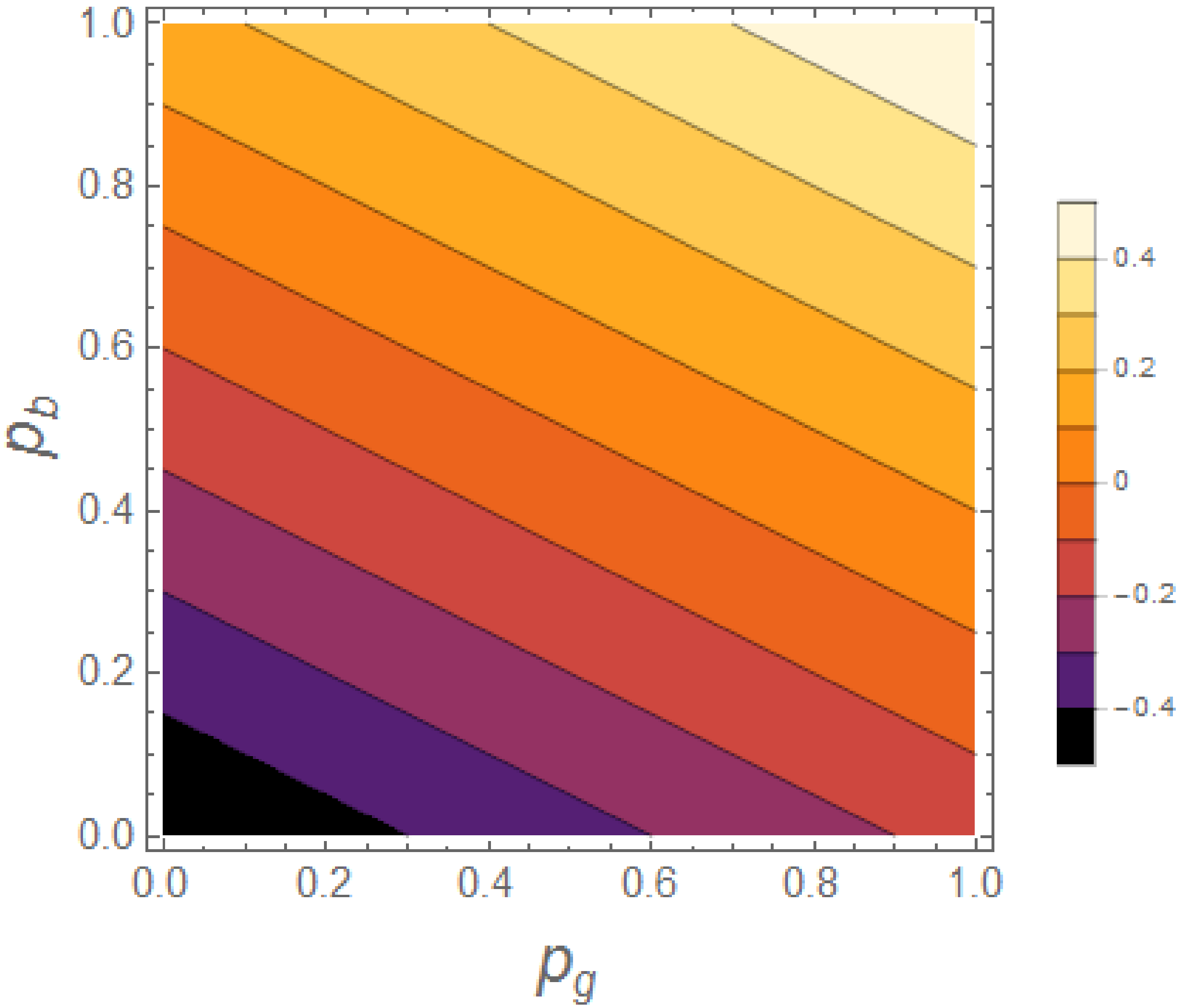
The Dependency on Module M



Optimal α for M=3



The Capital Gain at α=3/4 when M=3



The matrix formulation for two-time-steps memory

AA			AB			BA			BB		
0	(1 - a) p	(-1 + a) (-1 + p)	0	0	0	0	a p	a (1 - p)	0	0	0
(-1 + a) (-1 + p)	0	(1 - a) p	0	0	0	a (1 - p)	0	a p	0	0	0
(1 - a) p	(-1 + a) (-1 + p)	0	0	0	0	a p	a (1 - p)	0	0	0	0
0	(1 - b) p	(-1 + b) (-1 + p)	0	0	0	0	b p	b (1 - p)	0	0	0
(-1 + b) (-1 + p)	0	(1 - b) p	0	0	0	b (1 - p)	0	b p	0	0	0
(1 - b) p	(-1 + b) (-1 + p)	0	0	0	0	b p	b (1 - p)	0	0	0	0
0	0	0	0	c pb	c (1 - pb)	0	0	0	0	(1 - c) pb	(-1 + c) (-1 + pb)
0	0	0	c (1 - pg)	0	c pg	0	0	0	(-1 + c) (-1 + pg)	0	(1 - c) pg
0	0	0	c pg	c (1 - pg)	0	0	0	0	(1 - c) pg	(-1 + c) (-1 + pg)	0
0	0	0	0	d pb	d (1 - pb)	0	0	0	0	(1 - d) pb	(-1 + d) (-1 + pb)
0	0	0	d (1 - pg)	0	d pg	0	0	0	(-1 + d) (-1 + pg)	0	(1 - d) pg
0	0	0	d pg	d (1 - pg)	0	0	0	0	(1 - d) pg	(-1 + d) (-1 + pg)	0