Chronotaxic dynamics: when the characteristic frequencies fluctuate and the system is stable

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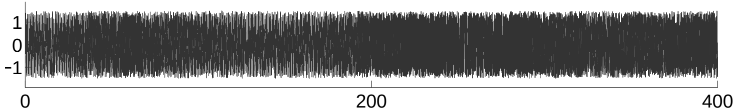
Outline

1. Unsolved problems
2. Non-autonomous dynamics: Chronotaxic systems
3. Reconstructing dynamics from time-series

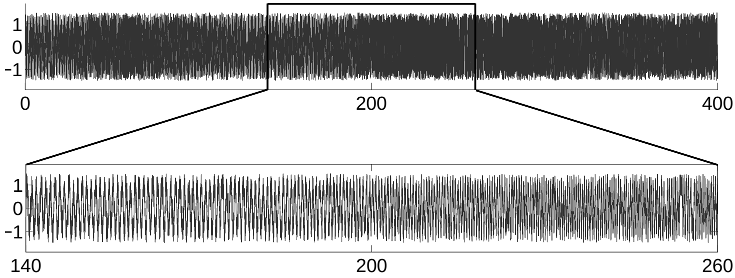
Unsolved problems

- Complex, fluctuating dynamics abounds in nature.
- Theoretical, or model-driven approach: Diffusion and flow in space and time; Energy and information.
- Data-driven approach: many insufficiently considered issues: non-stationarity, interactions and couplings, time-variability, finite vs infinite time, non-autonomically, complexity.
- **So, how to bridge the model-driven and the data-driven approaches?**

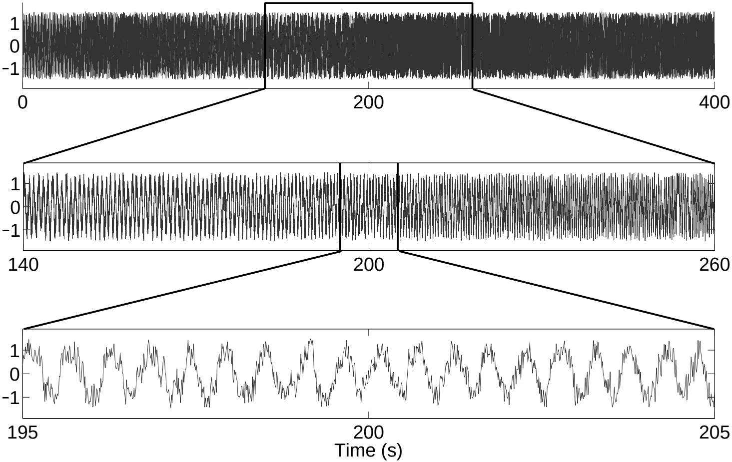
Is this noise?



Is this noise?



Is this noise?



Is this noise?

The wavelet transform may reveal clear evidence of determinism in signals which initially appear to be noisy.

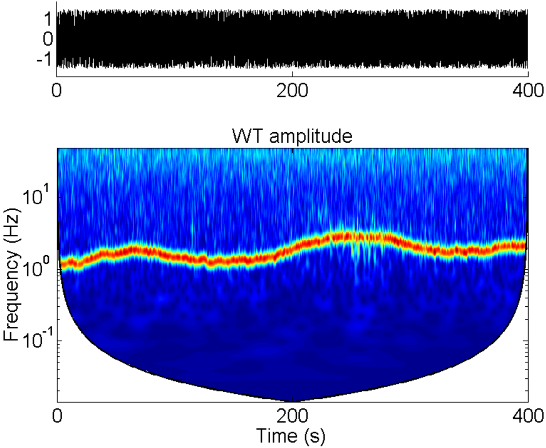
Data shown is $\sin(\alpha_{\mathbf{x}})$ from

$$\begin{aligned}\dot{\alpha_{\mathbf{p}}} &= \omega_{\mathbf{p}}\\ \dot{\alpha_{\mathbf{x}}} &= \omega_{\mathbf{x}} - \varepsilon\sin(\alpha_{\mathbf{x}} - \alpha_{\mathbf{p}}) + \eta(t)\end{aligned}$$

where $\varepsilon = 15$, noise strength $=$ 2 and $\omega_{\mathbf{p}}$ varies according to

$$\begin{aligned}\dot{\omega}_{var}(t) &= Acos(2\pi f_{\mathbf{m}} t) + \eta(t)\\ \omega_{\mathbf{p}}(t) &= 2\pi f_p + \omega_{var},\end{aligned}$$

where $f_{\mathbf{m}}$ is the modulation frequency (0.005 Hz), f_p is the average frequency (1 Hz), A=0.1 and noise strength $=$ 0.2. Integration step $=$ 0.01



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Non-autonomous systems

Non-autonomous systems explicitly depend on time: the rhs of the corresponding differential equations depend on time.

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, ..., x_n, t) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, ..., x_n, t)
\end{aligned}
$$

Conventional method to consider such systems – to introduce a new variable $x_{n+1} = t$, thus

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, ..., x_n, x_{n+1}) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, ..., x_n, x_{n+1}) \\
\dot{x}_{n+1} &= 1
\end{aligned}
$$

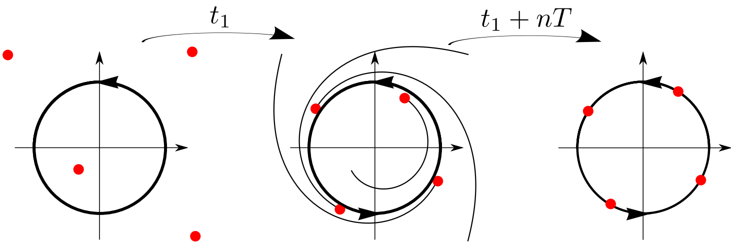
But often such an approach is not useful, and non-autonomous systems should be considered as non-autonomous.

Conventional autonomous models of oscillatory systems

The conventional theories are based on limit cycle oscillators, e.g. in polar coordinates (r, α)

$$\begin{aligned} \dot{r} &= -\varepsilon r\,(r - r_0)\,, \\ \dot{\alpha} &= \omega. \end{aligned}$$

They are described by a phase α with zero characteristic Lyapunov exponent.

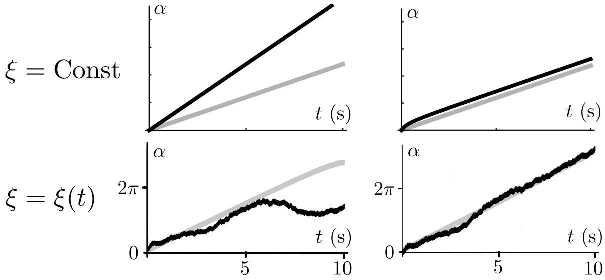


Limit cycle oscillators

Conventional limit cycle oscillators

- A phase shift does not decay and does not grow, it stays the same
- A phase can be easily perturbed by any external perturbation
- A frequency can be changed by smallest continuous perturbation

Conventional: $\dot{\alpha} = \omega + \xi$ **New:** $\dot{\alpha} = F(\alpha, t) + \xi$



Main characteristics of chronotaxic systems

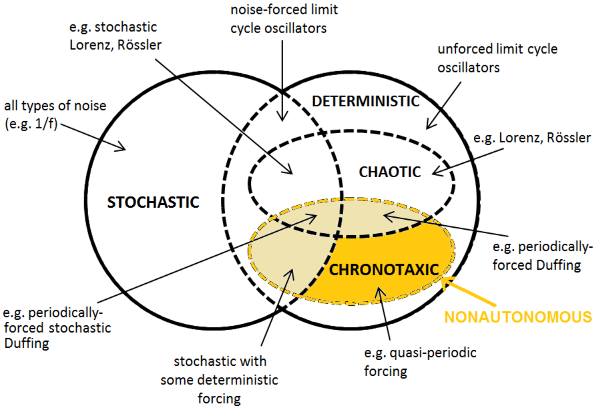
We therefore define a **new class** of non-autonomous oscillators:
chronotaxic systems (from *chronos* – time and *taxis* – order).

Suprunenko, Clemson and Stefanovska, *PRL* (2013); *PRE* (2014)

Their definition is based on following concepts –

1) Non-autonomous systems

2) Time-dependent point attractor (driven steady state)



Chronotaxic systems: Defining concepts

Time-dependent point attractor (driven steady state) $\mathbf{x}^A(t)$ in $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, t)$.

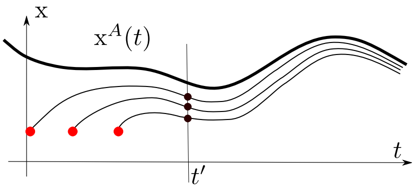
$\mathbf{x}^A(t)$ satisfies mathematical conditions of pullback attraction (1), forward attraction (2) and invariance (3):

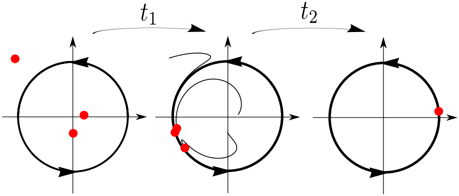
$$\lim_{t_0 \to -\infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = \mathbf{x}^A(t)$$
$$\lim_{t \to \infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = \mathbf{x}^A(t)$$
$$\mathbf{x}(t, t_0, \mathbf{x}^A(t_0)) = \mathbf{x}^A(t)$$

Kloeden and Rasmussen, Nonautonomous Dynamical Systems (2011)

Attraction at all times: deviations only decay, $\delta\mathbf{x}^A(t) = \mathbf{x}(t) - \mathbf{x}^A(t)$,

$$\frac{d}{dt}|\delta\mathbf{x}^A|^2 = 2\delta\mathbf{x}^{A\ T} J(\mathbf{x}^A, t)\delta\mathbf{x}^A < 0 \quad \text{and} \quad J(\mathbf{x}, t) = \frac{1}{2}\left(\partial\mathbf{g}/\partial\mathbf{x} + \partial\mathbf{g}/\partial\mathbf{x}^T\right)$$





Chronotaxic system in brief

A chronotaxic system is a non-autonomous oscillatory dynamical system $\mathbf{x}$ generated by an autonomous system of unidirectionally coupled equations

$$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}), \quad \dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{p}), \tag{1}$$

where $\mathbf{p} \in R^n$, $\mathbf{x} \in R^m$, $\mathbf{f} : R^n \to R^n$, $\mathbf{g} : R^m \times R^n \to R^m$; n and m can be any positive integers.

The system (1) is also called a drive and response system (Kocarev, 1996), or a master-slave configuration (Haken, 2004).

The solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$ of Eqs. (1), depends on the actual time t as well as on the initial conditions $(t_0, \mathbf{x}_0)$; whereas the solution $\mathbf{p}(t - t_0, \mathbf{p}_0)$ depends only on initial condition p_0 and on the time of evolution $t - t_0$.

The subsystem $\mathbf{x}$ is nonautonomous in the sense that it can be described by an equation which depends on time explicitly, e.g. $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{p}(t))$.

Chronotaxic properties from data

A chronotaxic system is described by $\mathbf{x}$ which is assumed to be observable, and $\mathbf{p}$ which may be inaccessible for observation, as often occurs when studying real systems.

Rather than assuming or approximating the dynamics of $\mathbf{p}$; we focus on the dynamics of $\mathbf{x}$ and use only the following simple assumption:
the system $\mathbf{p}$ is such that it creates a time-dependent steady state in the dynamics of $\mathbf{x}$.

Chronotaxic properties from data

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the system p is such that it creates a time-dependent steady state in the dynamics of x.

Hence, the whole external environment with respect to **x** is divided into two parts

- The part **p** which makes the system **x** chronotaxic.
- The second part contains the rest of the environment and is therefore considered as external perturbations.

The theory for the case where amplitudes and phases are separable have been introduced by Suprunenko, Clemson, Stefanovska, *PRL* (2013), *PRE* (2013).
Subsequently expanded to include the generalized case of chronotaxic systems where such decoupling is not required Suprunenko and Stefanovska, *PRE* (2014).

Benefits?

- When perturbations do not destroy the chronotaxic properties of a system, the stable deterministic component of its dynamics can be identified (Clemson, Suprunenko, Stankovski, Stefanovska, *PRE* (2013), Lancaster, Suprunenko, Clemson, Stefanovska, *Entropy* (2015)).

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- This reduces the complexity of the system, enabling us to filter out the stochastic component and focus on the deterministic dynamics and the interactions between system $\mathbf{x}$ and its driver $\mathbf{p}$.
- For complex and open systems it has the potential to extract dynamical properties of the system which were previously neglected, or considered as noise.
- The theory of chronotaxic systems could facilitate more realistic insight into the underlying dynamics of systems whose time-evolution is recorded.

An example of chronotaxic phase oscillator system

$$\begin{cases} \dot{\alpha}_{\mathbf{p}} = \omega_0(t) \\ \dot{\alpha}_{\mathbf{x}} = \varepsilon\omega_0(t)\sin(\alpha_{\mathbf{x}} - \alpha_{\mathbf{p}}) + \xi(t) \end{cases}$$

ε is the coupling strength from the external variable **p** to the observed variable **x**.

The function $\xi(t)$ is white Gaussian noise with standard deviation $\eta = \sqrt{2E}$, where $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(\tau)\rangle = \delta(t-\tau)E$.

The frequency of $\alpha_{\mathbf{p}}$ is given as

$$\omega_0(t) = \omega_1\left[1 + A\sin(\omega_2 t)\right].$$

Three chronotaxic modes

mode 1: $[\omega_1 = 2\pi,\ \omega_2 = 0.016\pi,\ A = 1]$
mode 2: $[\omega_1 = 0.3\pi,\ \omega_2 = 0.005\pi,\ A = 0.15]$
mode 3: $[\omega_1 = 0.05\pi,\ \omega_2 = 0.001\pi,\ A = 0.025]$
In each case $|\varepsilon| = 1.5$.

Example time series $x(t)$

$$x(t) = \cos(\alpha_{\mathbf{x},1} t) + \cos(\alpha_{\mathbf{x},2} t) + \cos(\alpha_{\mathbf{x},3} t) + \eta_1(t), \tag{2}$$

where $\alpha_{\mathbf{x},i}$ are the phases of each of the modes and $\eta_1(t)$ is a 1/f noise signal.

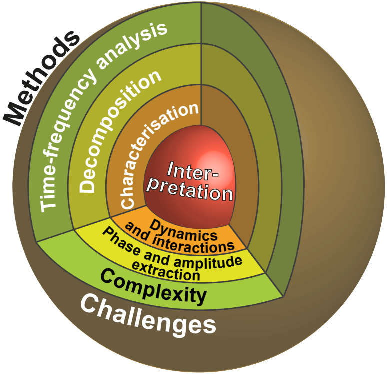
A second time series $p(t)$, containing the external modes which drive the **x**, is

$$p(t) = \cos(\alpha_{\mathbf{p},1} t) + \cos(\alpha_{\mathbf{p},2} t) + \cos(\alpha_{\mathbf{p},3} t) + \eta_2(t),$$

where $\alpha_{\mathbf{p},i}$ are the phases of the modes driving the system and $\eta_2(t)$ is a separate 1/f noise signal.

Outline

Challenges in time-series analysis

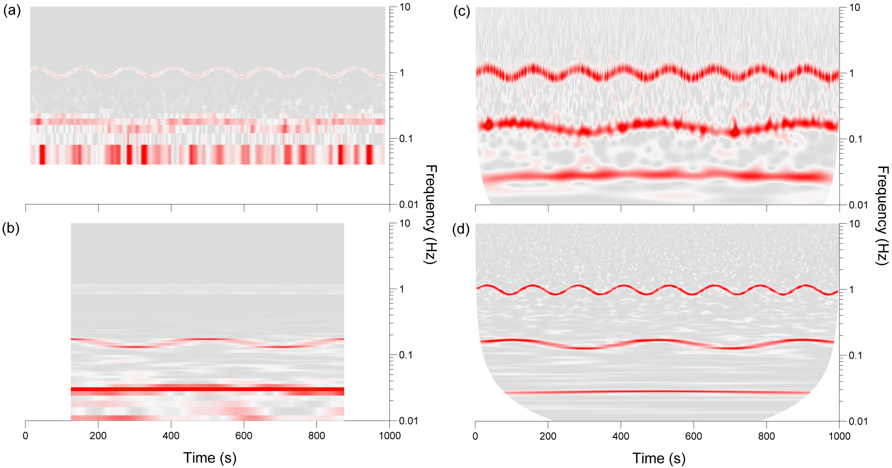


An illustration showing the challenges related to each level of analysis and the corresponding methods used to tackle them.

- Time-frequency analysis
- Identification and extraction of the phase and amplitude of individual oscillatory components using decomposition methods
- Characterisation of the dynamics of the modes and detection how they interact with each other
- Given this information the properties of an explicit physical model of the dynamics provide an interpretation of the system that generated the signal.

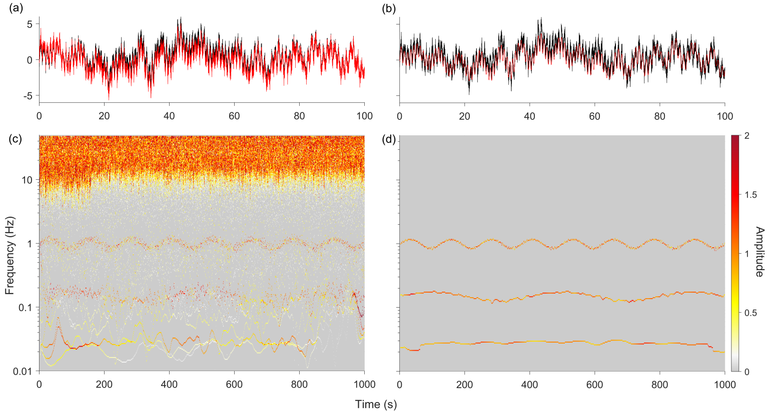
This and the subsequent figures (slides 18–25) are taken from the manuscript "Reconstructing Time-Dependent Dynamics", P. Clemson, G. Lancaster and A. Stefanovska, which has been accepted for publication in the *Proceedings of the IEEE* on 2 October, and will be published in early 2016.

Optimalisation of the uncertainty principle in time-frequency analysis



- STFT and continuous Morlet wavelet transforms of the time series defined by Eq. (2).
- (a) and (b) STFTs for a 25 s and 250 s window respectively.
- (c) and (d) Continuous Morlet wavelet transforms with the central frequencies $f_0 = 1$ and $f_0 = 5$ respectively.
- White spaces indicate the limit of the cone of influence where the transform is not defined.

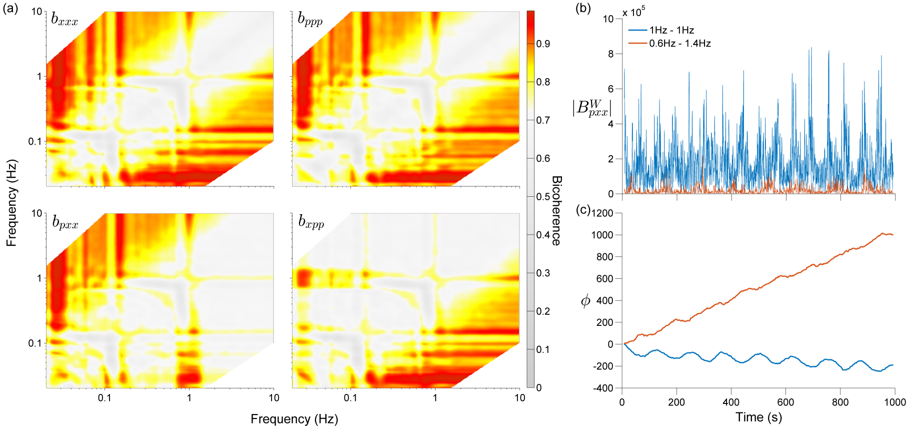
Extraction of a few meaningful modes vs "complete" decomposition of meaningless modes



- Decomposition of the time series defined in Eq. (2)
- (a) The signal reconstructed from the first 19 IMFs from EMD is shown by the red line
- (b) Reconstruction from NMD is shown by the red line
- In both plots the black line corresponds to the original signal
- (c) and (d) The amplitude of the modes transformed to the time-frequency domain using the Hilbert transform[1] for EMD and NMD respectively.

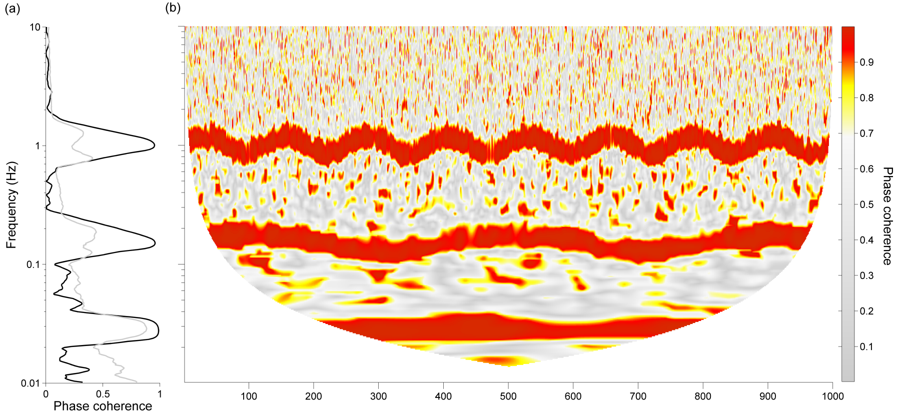
[1] The Hilbert transform generates the analytic signal of a sinusoidal oscillation, which can then be used to calculate its instantaneous frequency. This offers a direct comparison with the results of time-frequency analysis.

Tracking couplings between oscillations in time



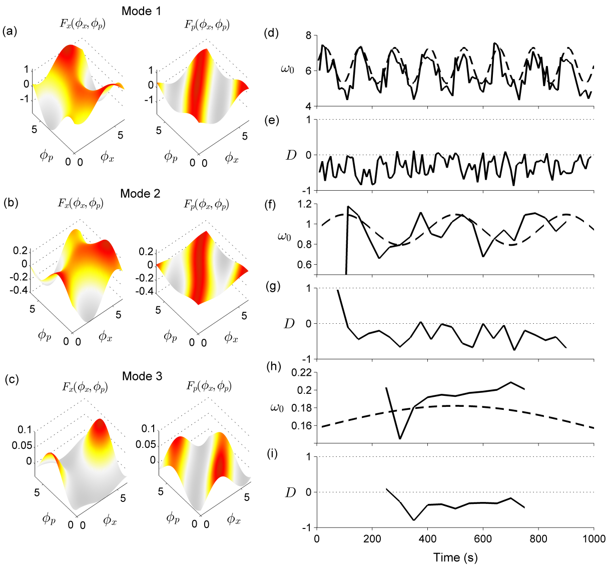
- Wavelet bispectrum analysis for the time series $x(t)$ and $p(t)$.
- (a) The bicoherence for the various combinations of the cross-bispectrum. Without the time axis it is difficult to distinguish the amplitude due to noise fluctuations from the amplitude contributions of genuine couplings.
- (b) and (c) The bispectral amplitude and phase for the coupled frequency pair (1 Hz, 1 Hz) and unrelated frequency pair (0.6 Hz, 1.4 Hz). While the amplitude is higher for the coupled pair, the coupling is also indicated by the fact that the phase does not grow over time.

Detecting common time-varying components in different time series



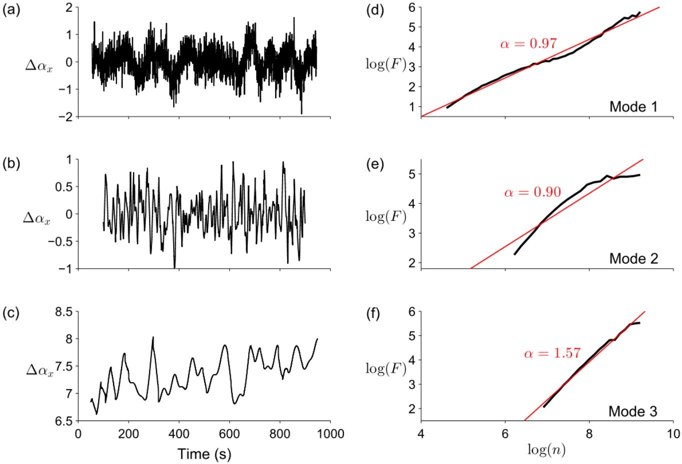
- Wavelet phase coherence between the time series $x(t)$ and $p(t)$.
- (a) Significant phase coherence is when the coherence (black line) is greater than the 95th percentile of 100 pairs of IAAFT surrogates (grey line).
- (b) Windowed phase coherence. It reveals the time-variability of the modes but at the cost of losing information about lower frequencies.

Inferring the coupling functions from the time series of two interacting systems



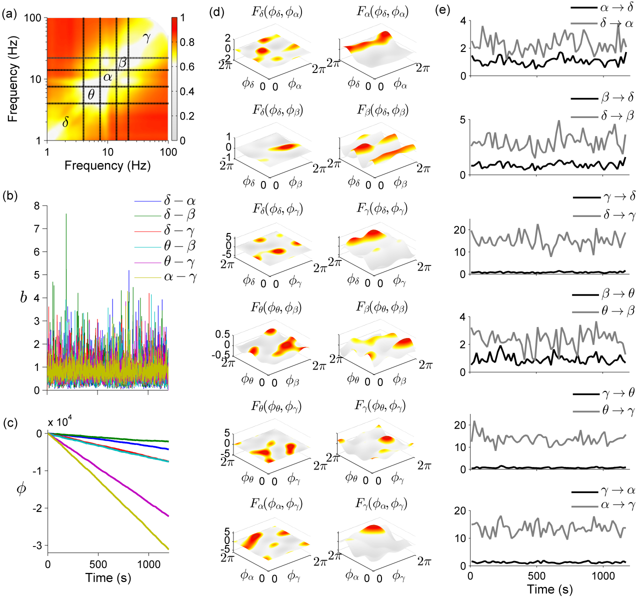
- Dynamical Bayesian inference analysis of the phases extracted using NMD from the time series $x(t)$ and $p(t)$.
- (a), (b) and (c) The coupling functions for the pairs of phases extracted from each mode.
- (d), (f) and (h) The inferred value of $\omega_0(t)$ (solid black line) and the actual value (dotted line).
- (e), (g) and (i) The direction of coupling calculated by taking the ratio of the amplitudes for the other terms dependent on the other phase for each coupling function. $D > 0$ for a coupling in the direction $x \longrightarrow p$ and $D < 0$ for a coupling in the direction $p \longrightarrow x$.
- The model parameters were inferred using a 20 s moving window with 50% overlap for mode 1, a 150 s window with 75% overlap for mode 2 and a 500 s window with 90% overlap for mode 3.
- In each case the propagation constant had the value $p = 0.2$

Measuring the strength of chronotaxicity from the fractal nature of the observed perturbations



- Identifying *chronotaxicity* in the data provides information about the underlying dynamics.
- Chronotaxicity can be identified by examining phase fluctuations of the oscillatory mode.
- In a non-chronotaxic system, these fluctuations will resemble a random walk, whilst in a chronotaxic system they will appear closer to white noise.

Bispectral and dynamical Bayesian inference of EEG



- The signal was measured for 20 minutes from the forehead of a subject in anæsthesia.
- Phases for the δ (0.8-4 Hz), θ (4-7.5 Hz), α (7.5-14 Hz), β (14-22 Hz) and γ (22-100 Hz) waves, extracted using NMD.
- (a) Bicoherence of the raw EEG signal.
- (b) and (c) Instantaneous bicoherence and phase of the bispectrum respectively for the pairs of brain waves.
- (d) Coupling functions for the different pairs of extracted phases are shown (couplings between adjacent bands are not shown due to frequency spillage from imperfect filtering).
- (e) The magnitude of the coupling functions for each point in time, providing an indication of the direction of coupling between the phases.
- The model parameters were inferred using a 20 s moving window with no overlap and with the propagation constant $p = 0.2$.

More unsolved problems

- How to detect chronotaxicity in systems where the amplitude and phase are not separable, as e.g. is the case of brain dynamics? This will allow for studiyng amplitude-amplitude and amplitude-phase interactions, in addition to the phase-phase dynamics.
- How to include the spatial dynamics in addition to the current theory of temporal dynamics? This will further widen the applicability, e.g. to cellular dynamics,

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- **Many thanks for your attention.**