Scaling and Rare Events near Excitation Threshold of a Parametric Oscillator

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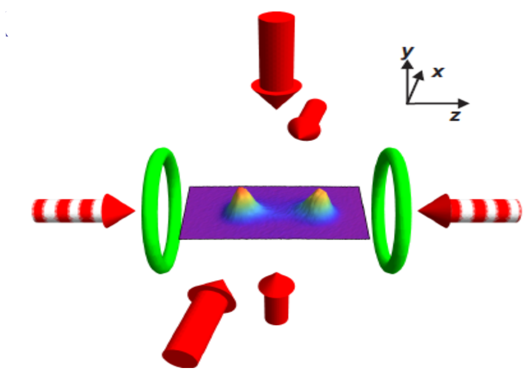
Eigenstates of a periodically driven system are not stationary:

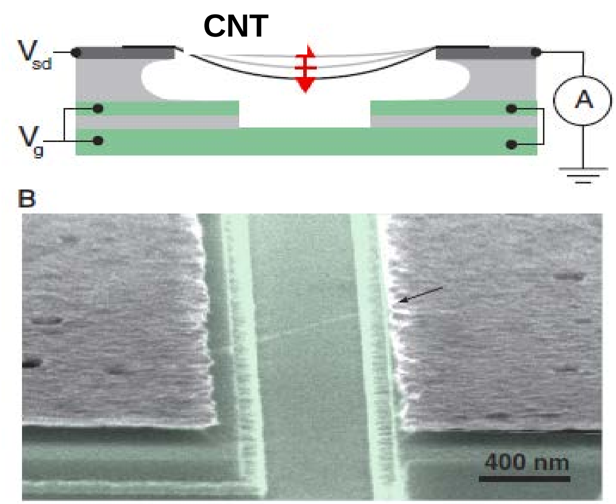
$$H(t) = H_0(q,p) - qF\cos\omega_F t, \qquad i\hbar\dot{\psi} = H(t)\psi$$

$$\psi_\varepsilon(t) = e^{-i\varepsilon t/\hbar}\, u_\varepsilon(t), \qquad u_\varepsilon\left(t + \frac{2\pi}{\omega_F}\right) = u_\varepsilon(t)$$

quasienergy ≡ Floquet eigenvalue; quantization: $\varepsilon \to \varepsilon_n$

Driven mesoscopic vibrational systems of current interest: Josephson junctions, cavity modes in optical and superconducting cavities, nanomechanical systems, cold atoms,…





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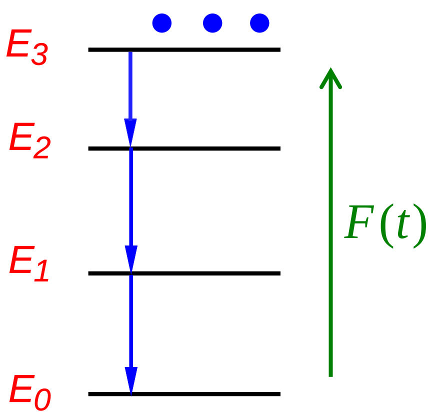
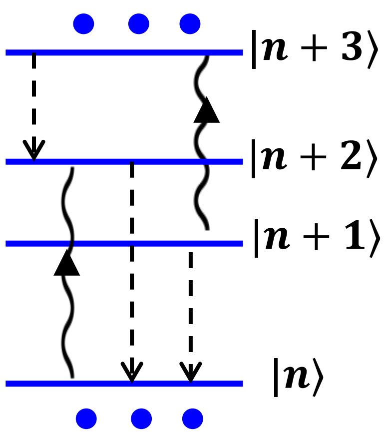
Relaxation, $T = 0$: inter-state transitions with emission of photons, phonons, etc.

Fock states



Quasienergy states are linear combinations of Fock states. Inter-level transitions down in energy, $|N_{Fock}\rangle \to |N_{Fock} - 1\rangle$, correspond to inter-quasi-energy level transitions $|n\rangle \to |n \pm m\rangle$, "up" and "down" in quasienergy. Even where the energy-level width $\Gamma \ll \Delta E$, we can have $\Gamma \geq \Delta\varepsilon$

Quasienergy states

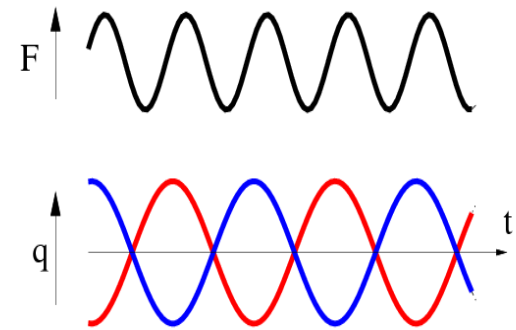


Problems: distribution over the quasienergy states? Effects of the breaking of the discrete-time symmetry? Related features of quantum fluctuations?

Classical phenomenological description, $m = 1$:

$$\ddot{q} + 2\Gamma\dot{q} + (\omega_0^2 + F\cos\omega_F t)q + \gamma q^3 = 0$$

Weak damping, resonant modulation $\omega_F \approx 2\omega_0 \Rightarrow$ excitation for weak field, small nonlinearity. The period-two states differ in phase by π - spontaneous breaking of discrete time-translation symmetry



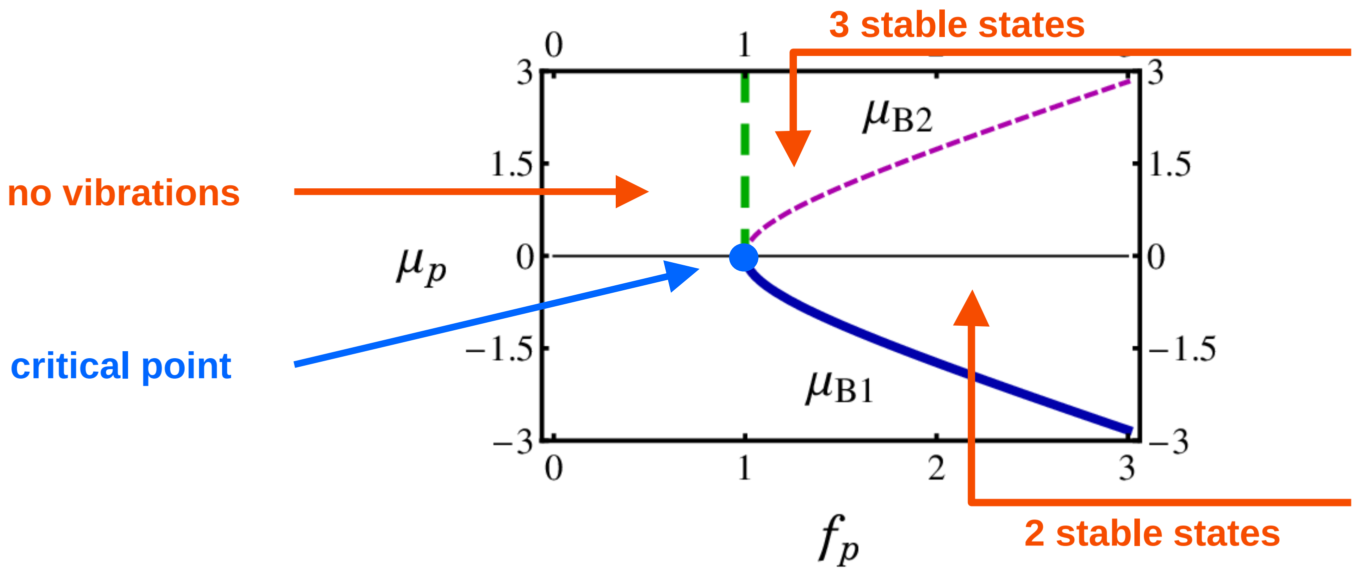
$$\ddot{q} + 2\Gamma\dot{q} + (\omega_0^2 + F\cos\omega_F t)q + \gamma q^3 = 0$$

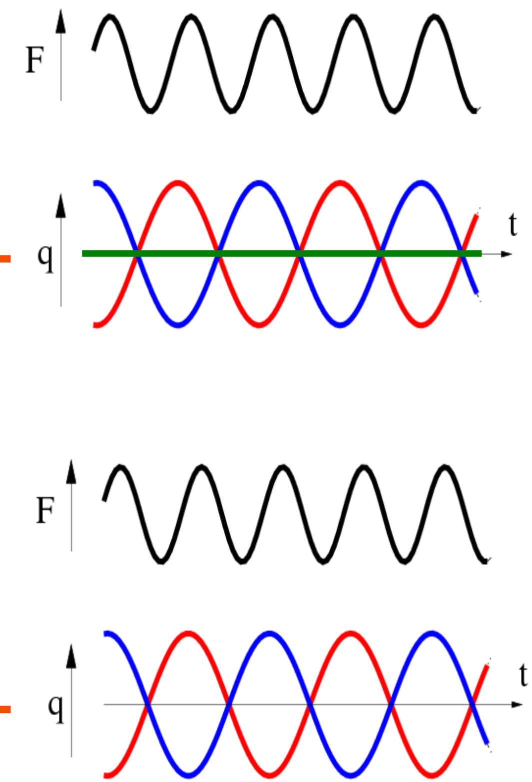
Critical field strength: $F_c = 2\Gamma\omega_F, \;\; F_c \ll \omega_0^2$

Relevant dimensionless parameters:

Scaled frequency detuning $\mu_p = (\omega_F - 2\omega_0)/2\Gamma$

Scaled field amplitude $f_p = F/F_c$





more complicated than just symmetry-breaking

co-dimension 2 bifurcation point

The rotating wave approximation (RWA)

Change to variables that slowly vary over the vibration period: ⬅ $\ddot{q}+2\Gamma\dot{q}+(\omega_0^2+F\cos\omega_F t)q+\gamma q^3=0$

$$q(t)=C(Q\cos\phi+P\sin\phi),\quad p(t)=-\tfrac{1}{2}\omega_F C(Q\sin\phi-P\cos\phi),\quad \phi=\tfrac{1}{2}\omega_F t+\tfrac{1}{4}\pi;$$

Quantum mechanics: $[p,q]=-i\hbar \quad\rightarrow [P,Q]=-i\,\tilde{\hbar},\quad \tilde{\hbar}=3|\gamma|\hbar/\omega_F F_c$

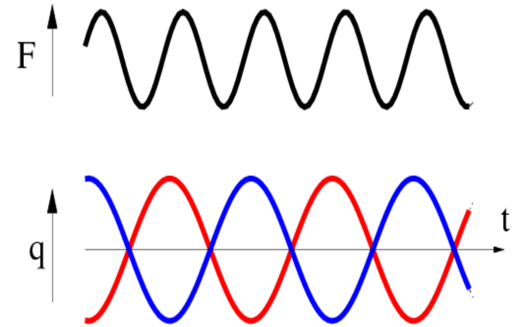
dimensionless Planck constant

Approximations: **slow decay,** $\Gamma\ll\omega_0$**, + weak quantum noise,** $\tilde{\hbar}\ll 1$

depends on the nonlinearity!

Quantum Langevin equations

In slow time



$$\dot{Q} = -\frac{i}{\tilde{\hbar}}[Q, g] - Q + \xi_Q(\tau), \quad \dot{P} = -\frac{i}{\tilde{\hbar}}[P, g] - P + \xi_P(\tau)$$

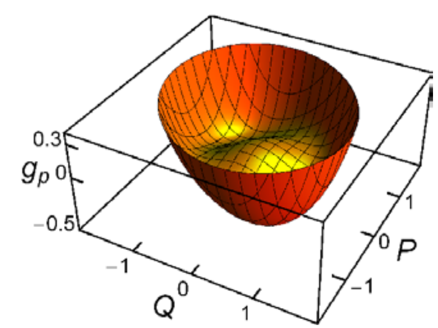
Quantum noise is δ-correlated in slow time:

$$\langle \xi_Q(\tau)\xi_Q(\tau')\rangle = \langle \xi_P(\tau)\xi_P(\tau')\rangle = 2D\delta(\tau - \tau')$$

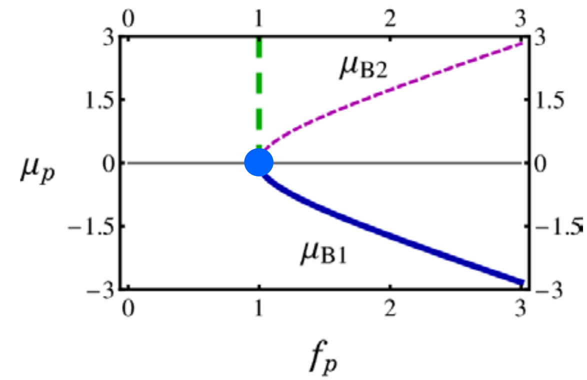
$$D = \tilde{\hbar}\left(\bar{n} + \frac{1}{2}\right), \bar{n} = \left(e^{\hbar\omega_0/k_B T} - 1\right)^{-1}, \qquad \langle[\xi_Q(\tau), \xi_P(\tau')]\rangle = 2i\tilde{\hbar}\,\delta(\tau - \tau')$$

Noise intensity $D \propto \hbar$ for $k_B T < \hbar\omega_0$; for $k_B T \gg \hbar\omega_0, D \propto T$

$$g(Q,P) = \frac{1}{4}\left(Q^2 + P^2\right)^2 - \frac{1}{2}\mu_p\left(Q^2 + P^2\right) + \frac{1}{2}f_p(QP + PQ)$$



$$\dot{Q} = -\frac{i}{\tilde{\hbar}}[Q, g] - Q + \xi_Q(\tau), \quad \dot{P} = -\frac{i}{\tilde{\hbar}}[P, g] - P + \xi_P(\tau)$$



Linear equations without noise near the critical point, $f_p = 1, \; \mu_p = 0$:

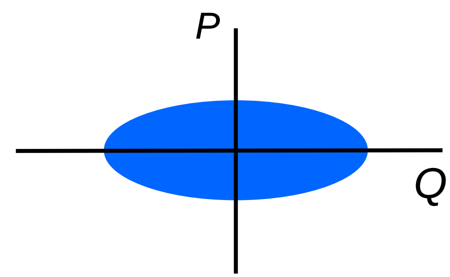
$$\dot{Q} \approx (f_p - 1)Q - \mu_p P, \qquad \dot{P} \approx -(f_p + 1)P + \mu_p Q$$

⬆

Q is a "soft mode"

$P(\tau)$ **adiabatically follows** $Q(\tau) \Rightarrow$ **on times** $\tau \gg 1$ $(\Gamma t \gg 1)$ **eliminate** $P(\tau) \Rightarrow$

an adiabatic classical equation for the soft mode with quantum noise

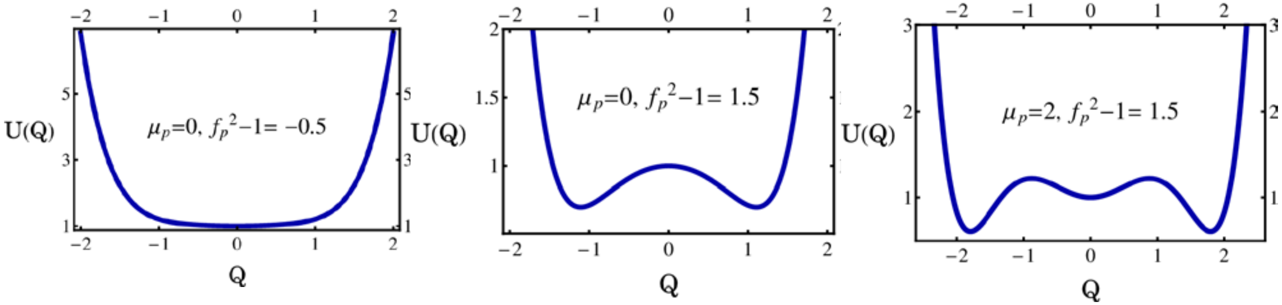


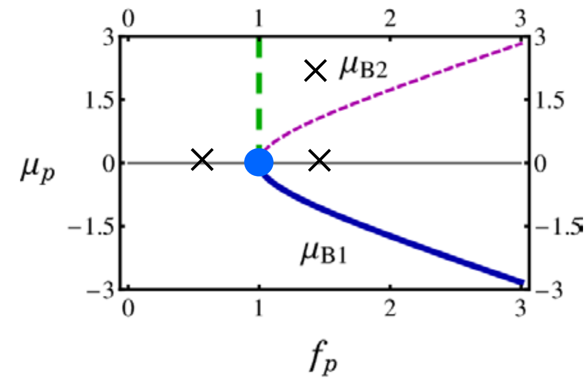
$$\dot{Q} = -\partial_Q U(q) + \xi_Q(\tau),$$

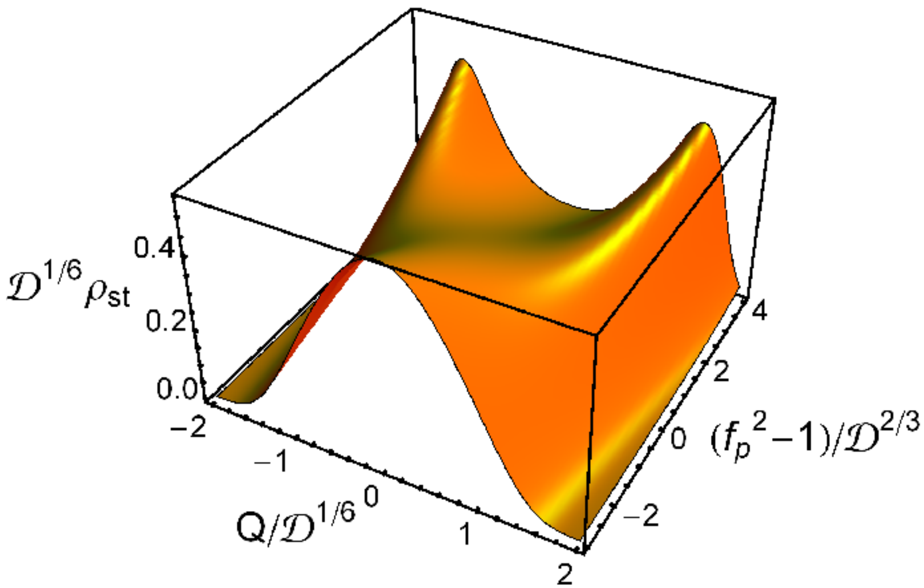
$$U(Q) = \frac{1}{4}\left[\mu_p^2 - \left(f_p^2 - 1\right)\right]Q^2 - \frac{\mu_p}{4}Q^4 + \frac{1}{12}Q^6$$

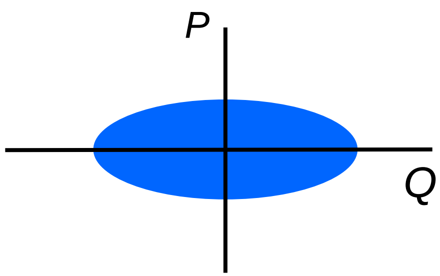
an analog of the ϕ^6 Landau theory

reminder: $f_p = F/F_c\,, \quad \mu_p \propto \omega_F - 2\omega_0$







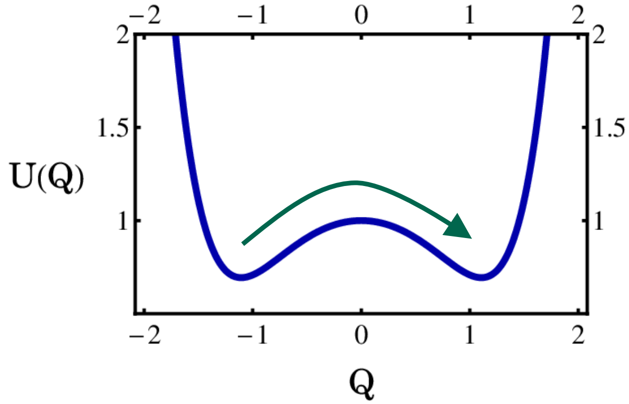


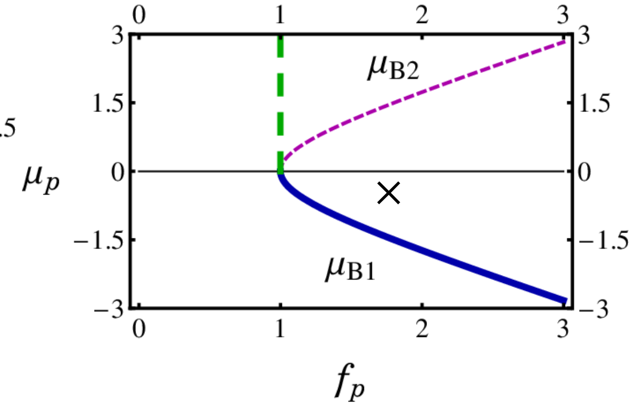
Critical region: the typical scales are $\Delta Q \sim D^{1/6} \propto \hbar^{1/6}$, $\Delta f_p \sim D^{2/3}$, $\Delta \mu_p \sim D^{1/3}$

The Wigner distribution $\rho_W(Q,P) \propto \exp\{-[P-P_{ad}(Q)]^2/2(f_p+1)D\} \exp[-U(Q)/D]$

Switching between period-two states

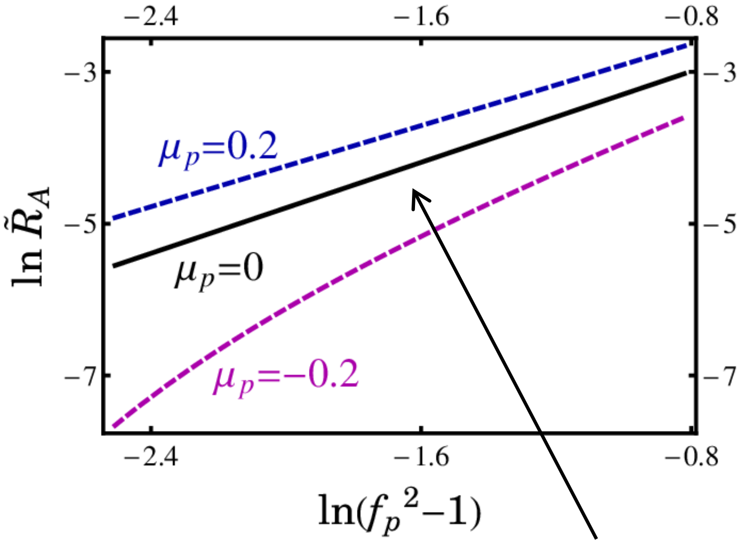
in the range of developed bistability





$$W_{sw} = \Omega_{sw} \exp\left(-\tilde{R}_A/\tilde{\hbar}\right),$$

$$\tilde{R}_A = \Delta U/(\bar{n} + \frac{1}{2})$$



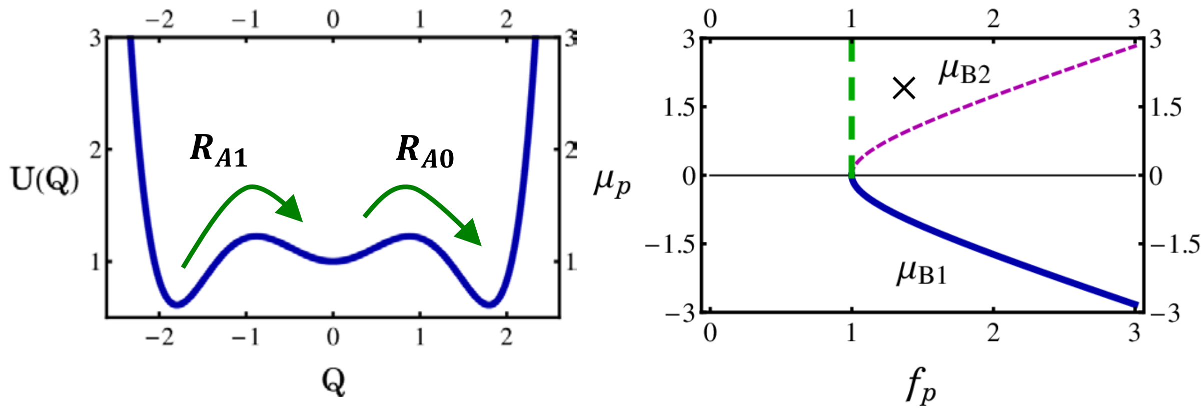
simple power-law scaling only for $\mu_p = 0$

(exact resonance, $\omega_F = 2\omega_0$)

$$\tilde{R}_A \propto \left(f_p^2 - 1\right)^{3/2}$$

Switching between period-two states

in the range of developed bistability



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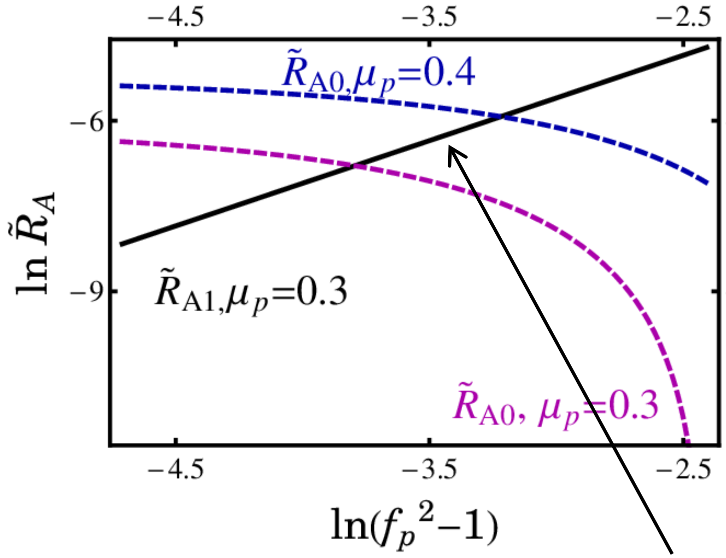
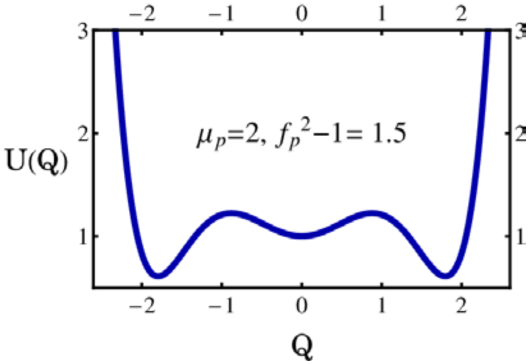
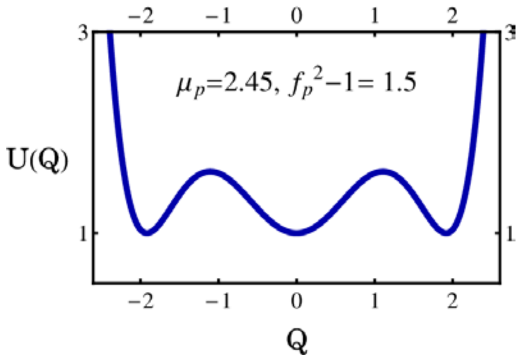
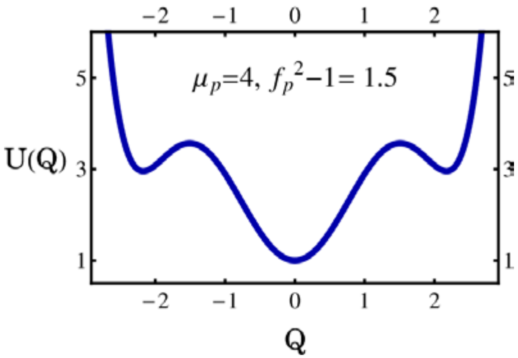


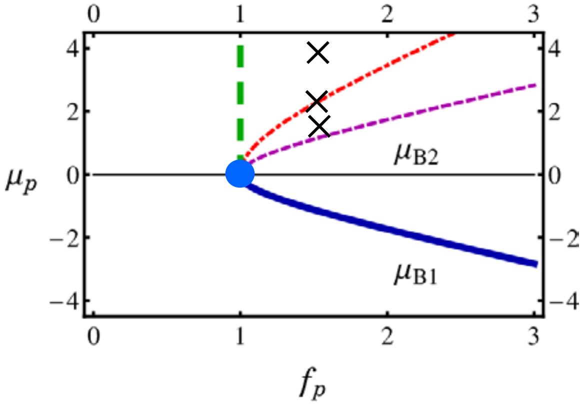
$\tilde{R}_{A1} \propto \left(f_p^2 - 1\right)^{3/2}$ **independent of** μ_p,

i.e. of the driving frequency detuning





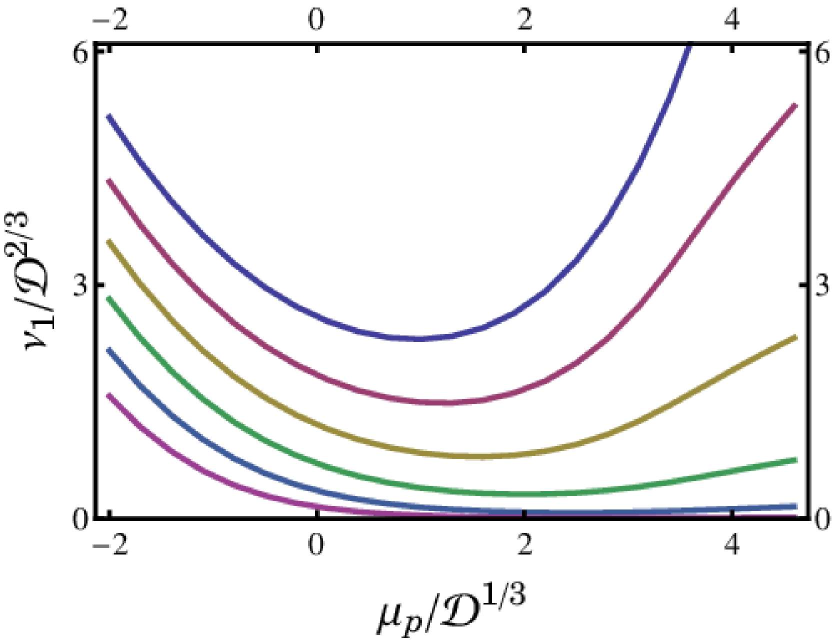


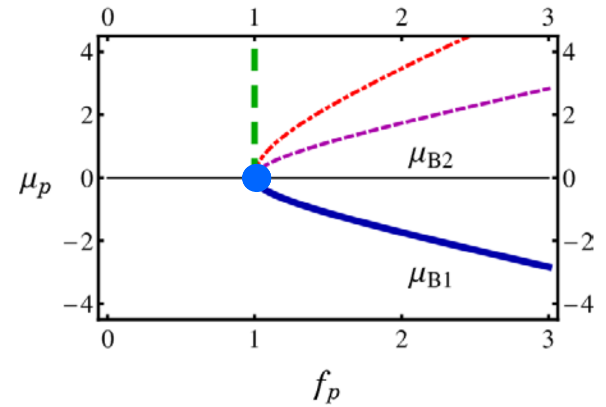


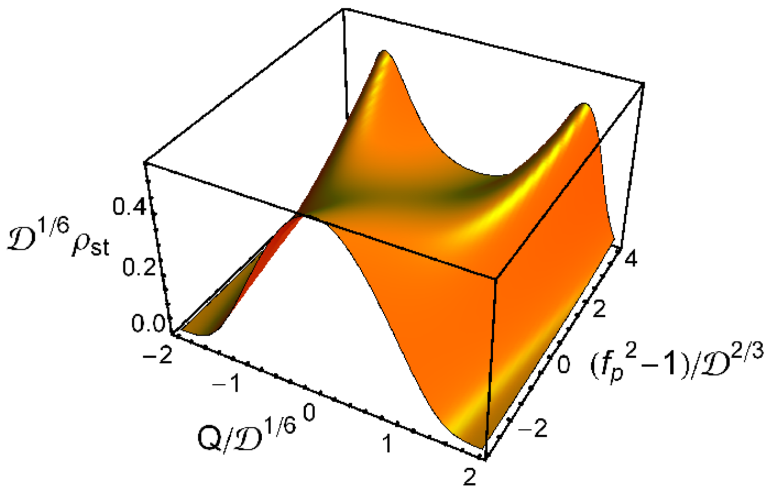
$$\mu_p^{cr} = 2\left(f_p^2 - 1\right)^{1/2}$$

Critical slowing down

Critical region: the typical scales are $\Delta Q \sim D^{1/6} \propto \hbar^{1/6}$,
$\Delta f_p \sim D^{2/3}$, $\Delta\mu_p \sim D^{1/3}$, $\Delta\tau \sim D^{-2/3} \propto \hbar^{-2/3}$,

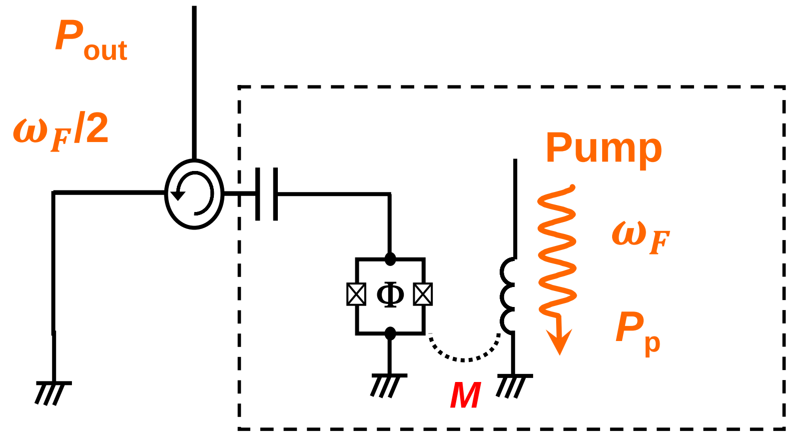






Reciprocal correlation time as function of the frequency detuning. From top down the scaled field is: $(f_p^2-1)/D^{2/3} = -4, -2, 0, 2, 4, 6.$

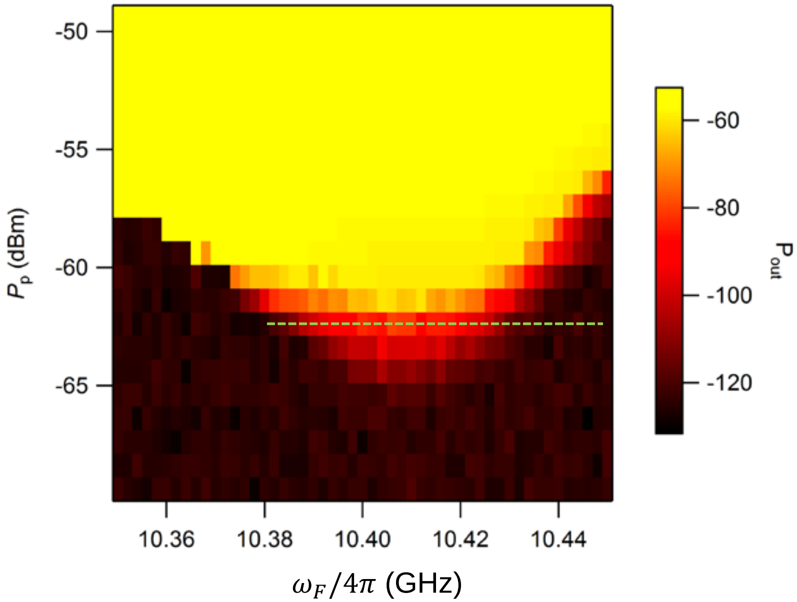
Schematics of the experimental system



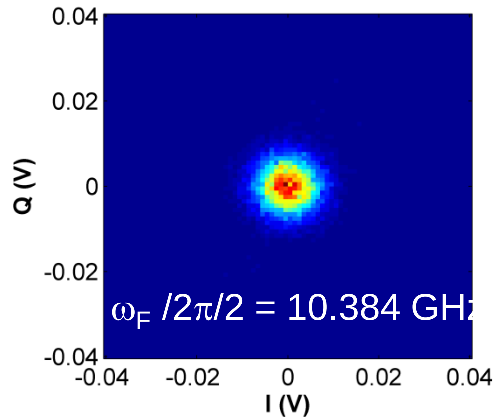
Temperature: T ~ 10 mK

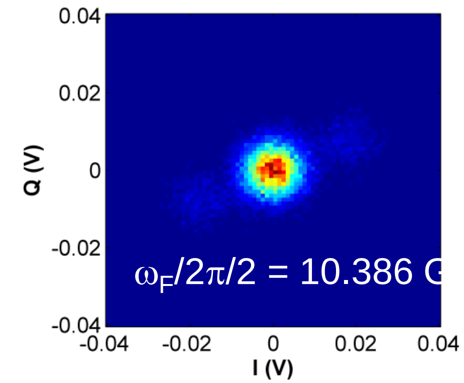
$\omega_0/2\pi$ = 10.402GHz, Q=340

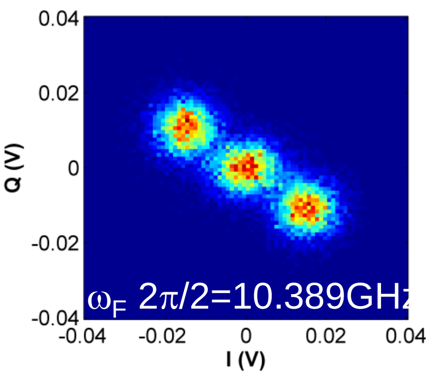
Vibrational states as a function of driving frequency

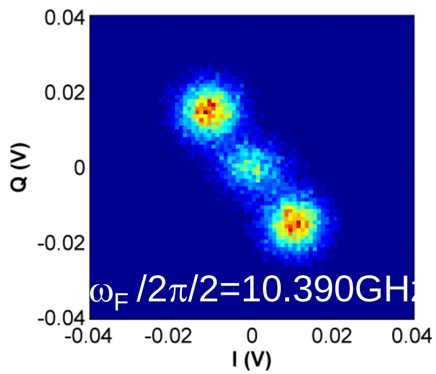


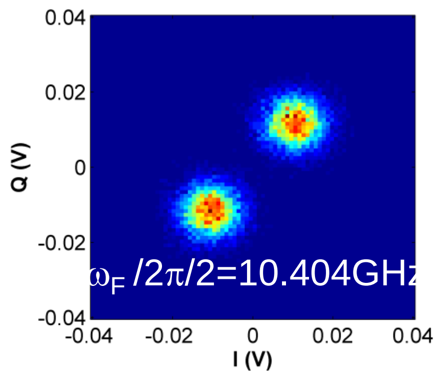
P_p = -62.4 dBm

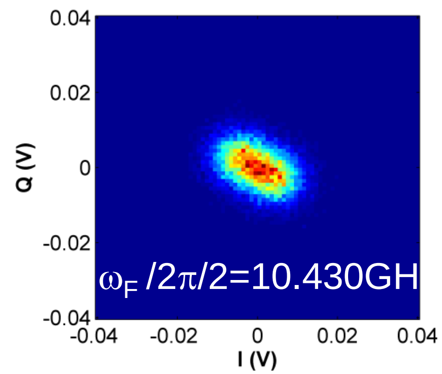




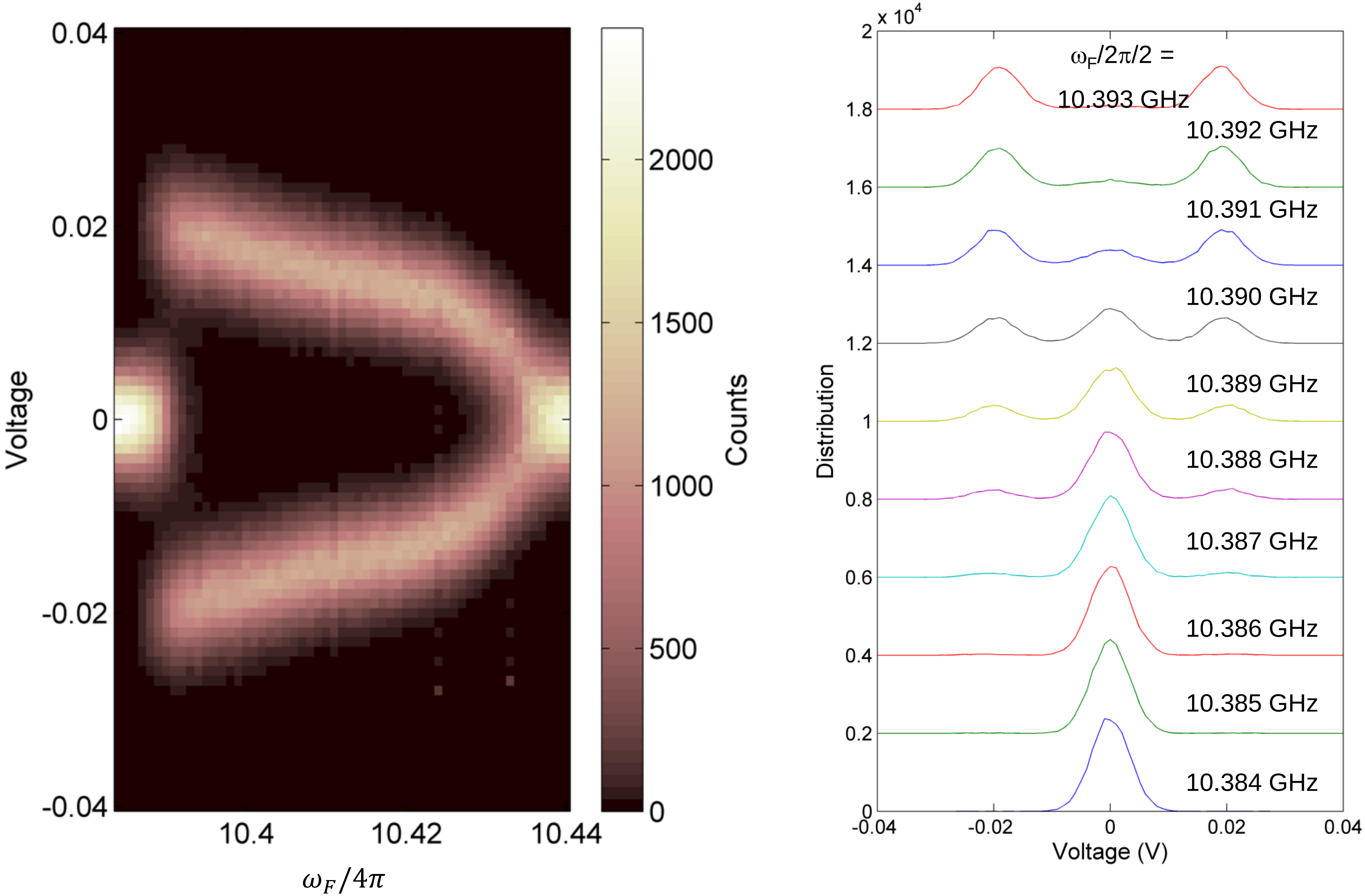






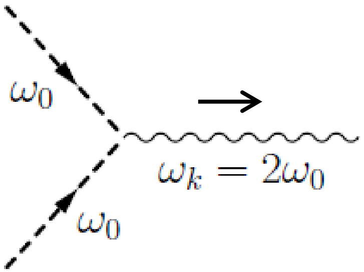


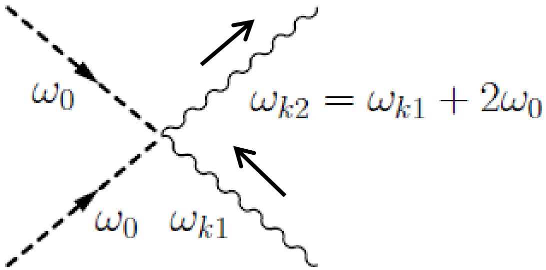
“First order phase transition”

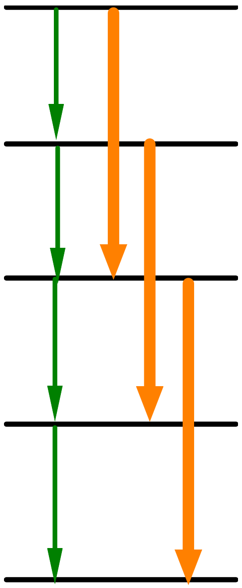


Phenomenological nonlinear friction: $f_{nl} = -2\Gamma_{nl} q^2 dq/dt$

A microscopic mechanism for passive quantum vibrational systems:







MD & Krivoglaz, 1975

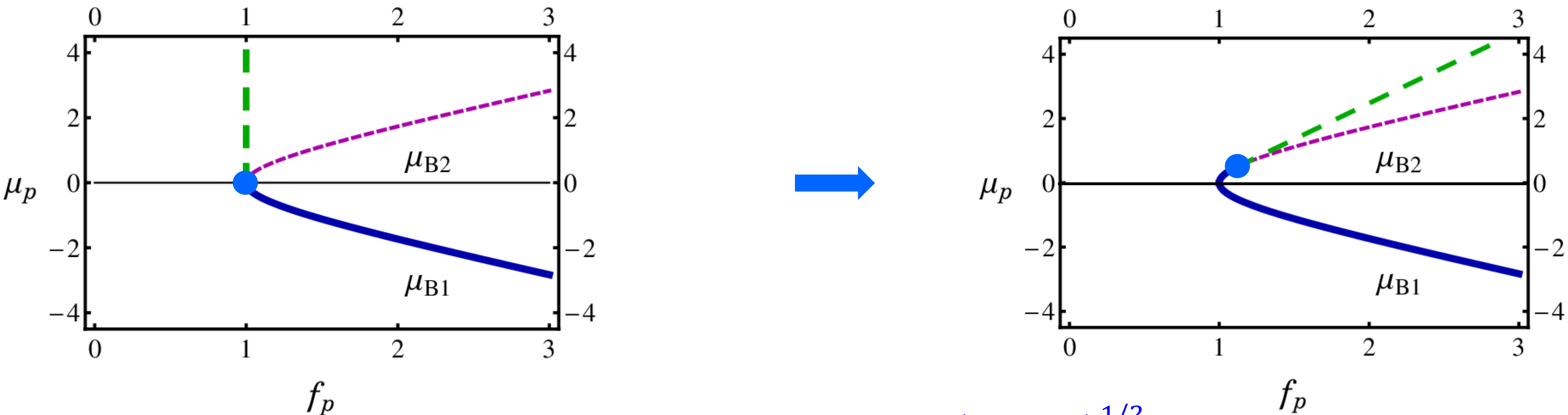
nanomechanics: Atalaya & MD, 2015

important for quantum optomechanics (MD, 1978)

Phenomenological nonlinear friction: $f_{nl} = -2\Gamma_{nl} q^2 dq/dt$

Quantum Langevin equations

$$\dot{Q} = -\frac{i}{\tilde{\hbar}}[Q, g_p] - Q + \xi_Q(\tau) - \frac{1}{2}\tilde{\Gamma}_{nl}\{Q, Q^2 + P^2\}_+ + \xi_Q^{nl}(t),$$
$$\dot{P} = -\frac{i}{\tilde{\hbar}}[P, g_p] - P + \xi_P(\tau) - \frac{1}{2}\tilde{\Gamma}_{nl}\{P, Q^2 + P^2\}_+ + \xi_P^{nl}(t)$$



critical point: $\mu_{p0} = \tilde{\Gamma}_{nl},\ f_{p0} = \left(\mu_{p0}^2 + 1\right)^{1/2}$

ϕ^6-type theory for the slow variable q near the critical point, $U(q) = \frac{1}{2}A_2 q^2 + \frac{1}{4}A_4 q^4 + \frac{1}{6}A_6 q^6$

$$\tilde{\Gamma}_{nl} = C^2\Gamma_{nl}/4\Gamma,\ A_2 = \frac{\delta\mu_p^2}{2f_{p0}^2} - f_{p0}\,\delta f_p,\ A_4 = -f_{p0}^2\delta\mu_p,\ A_6 = f_{p0}^6/2;\ \ \delta f_p = f_p - f_{p0} - \mu_{p0}\delta\mu_p/f_{p0}$$

Conclusions

- Near the critical point, parametric oscillators display critical slowing down and anomalously strong quantum fluctuations. The time scale, the fluctuation strength, and the width of the critical region are determined by fractional powers of $\hbar$.

- Quantum dynamics near the critical point is described by a slow variable driven by quantum noise, with a potential of the ϕ^6-type, for linear and nonlinear friction.

- Along with the time-symmetry breaking transition, the system displays a smeared first-order transition where three stable states are equally populated