Stationary states in 2D systems driven by Lévy noises

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I. INTRODUCTION

The description of complex interactions with surrounding can be provided provided by the Langevin equation, which in the overdamped limit takes the common form

$$\dot{x}(t) = f(x) + \zeta(t), \tag{1}$$

where $f(x)$ is a deterministic force, while $\zeta(t)$ represents complex interactions of a test "particle" with its environment. Usually it is assumed that the noise is white and Gaussian. Here, however we use the more general type of noise: i.e. Lévy noise, which is still of the white type but it naturally leads to heavy-tailed power-law fluctuations.

Heavy tailed fluctuation have been observed in versatilities of models including physics, chemistry or biology[1,2], paleoclimatology[3] or economics[4] and epidemiology[5] to name a few. Observations of the so-called Lévy flights boosted the theory of random walks and noise induced phenomena into new directions[6,7] which involve examination of space fractional diffusion equation (Smoluchowski-Fokker-Planck equation) and stimulated development of more general theory[8,9].

The present work[10] addresses properties of stationary states in 2D systems driven by Lévy flights. The research performed here extends earlier studies of 1D systems[11–16] where analysis of symmetric and asymmetric Lévy flights in harmonic, superharmonic and subharmonic potentials have been presented.

II. MODEL

In 1D, a motion of an overdamped particle subject to the α-stable Lévy type noise is described by the Langevin equation

$$\frac{dx}{dt} = -V'(x) + \sigma\zeta_{\alpha,0}(t), \tag{2}$$

which can be rewritten as $dx = -V'(x)dt + \sigma dL_{\alpha,0}(t)$, where $L_{\alpha,0}(t)$ is a symmetric α-stable motion[8], which is the generalization of the Brownian motion (Wiener process) to the situation when increments of the process are distributed according to α-stable densities, i.e. densities with power law asymptotics $p(x) \propto |x|^{-(\alpha+1)}$ with $0 < \alpha < 2$. $\zeta_{\alpha,0}(t)$ represents a white α-stable noise which is a formal time derivative of the symmetric α-stable motion $L_{\alpha,0}(t)$. Eq. (2) is associated with the fractional Smoluchowski-Fokker-Planck equation[17,18]

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[V'(x)p(x,t)\right] + \sigma^\alpha \frac{\partial^\alpha p(x,t)}{\partial |x|^\alpha} \tag{3}$$

where $\frac{\partial^\alpha}{\partial|x|^\alpha} = -(-\Delta)^{\alpha/2}$ is the fractional Riesz-Weil derivative (laplacian) defined via its Fourier transform $\mathcal{F}\left[-(-\Delta)^{\alpha/2}p(x,t)\right] = -|k|^\alpha \mathcal{F}\left[p(x,t)\right]$. Contrary to systems driven by white Gaussian noise, for $\alpha < 2$, the stationary solutions for Eq. (3) are not of the Boltzmann-Gibbs type and exist for potential wells which are steep enough[15]. The exponent c characterizing a potential well $V(x) = |x|^c$ needs to be larger than $2-\alpha$, i.e. $c > 2-\alpha$. Otherwise, the potential well is not steep enough in order to produce a stationary state. The stationary states (if exist) have power-law asymptotics

$$p_{\mathrm{st}}(x) \propto |x|^{-(c+\alpha-1)} \tag{4}$$

determined by the stability index α and the exponent c characterizing steepness of the potential well[13,15,16]. For $c = 2$ when the stationary density is of the same type (except the scale parameter) as the α-stable distribution associated with the underlying noise[13], see Eq. (2).

Analytical formulas for stationary states for systems driven by α-stable noises with $\alpha < 2$ are known only in a very limited number of cases. For the quartic 1D Cauchy oscillator, i.e. $V(x) = \frac{1}{4}x^4$ with $\alpha = 1$, the stationary state of the fractional diffusion equation (3) is given by[13] $p_{\mathrm{st}}(x) = \sigma/[\pi(\sigma^{4/3} - \sigma^{2/3}x^2 + x^4)]$.

By analogy to 1D system[17,18], the bi-variate system is described by the following Langevin equation driven by the bi-variate α-stable Lévy type noise

$$\frac{d\boldsymbol{r}}{dt} = -\nabla V(\boldsymbol{r}) + \sigma\boldsymbol{\zeta}_\alpha(t). \tag{5}$$

Eq. (5) can be rewritten as $d\boldsymbol{r} = -\nabla V(\boldsymbol{r})dt + \sigma d\boldsymbol{L}_\alpha(t)$, where $\boldsymbol{L}_\alpha(t)$ is a bi-variate α-stable motion and $V(\boldsymbol{r})$ is an external potential.

Equation (5) is associated with the Smoluchowski-Fokker-Planck equation which has the general form

$$\frac{\partial p(\boldsymbol{r},t)}{\partial t} = \nabla\cdot\left[\nabla V(\boldsymbol{r})p(\boldsymbol{r},t)\right] + \sigma^\alpha \boldsymbol{\Xi} p(\boldsymbol{r},t), \tag{6}$$

where $\boldsymbol{\Xi}$ is the fractional operator due to bi-variate α-stable noise $\boldsymbol{\zeta}$, see Eq. (5). The diffusive term in Eq. (6) depends on the noise type.[9] For the bi-variate α-stable noise with the uniform spectral measure the fractional operator $\boldsymbol{\Xi} = -(-\Delta)^{\alpha/2}$, i.e. it is the fractional laplacian defined via the Fourier transform $\mathcal{F}\left[-(-\Delta)^{\alpha/2}p(\boldsymbol{r},t)\right] = -|\boldsymbol{k}|^\alpha \mathcal{F}\left[p(\boldsymbol{r},t)\right]$.

The main scope of current research is to check if stationary states for harmonic and quartic potentials subject to bi-variate α-stable noises exist and what are their shapes depending on a noise type and noise parameters.

For the harmonic potential $V(x,y) = \frac{1}{2}(x^2+y^2)$ and the uniform spectral measure the fractional diffusion equation (6) takes the form

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}(xp) + \frac{\partial}{\partial y}(yp) - (-\Delta)^{\alpha/2}p, \tag{7}$$

where $p = p(x, y, t)$. In the Fourier space Eq. (7) is equivalent to

$$\frac{\partial \hat{p}}{\partial t} = -k\frac{\partial \hat{p}}{\partial k} - l\frac{\partial \hat{p}}{\partial l} - (k^2 + l^2)^{\alpha/2}\hat{p}. \tag{8}$$

The characteristic function of the stationary density is

$$\hat{p} = \exp\left[-\frac{(k^2 + l^2)^{\alpha/2}}{\alpha}\right], \tag{9}$$

which is the characteristic function of the bi-variate α-stable density with the uniform spectral measure. Therefore, in 2D like in 1D, the stationary state of the harmonic 2D oscillator is the bi-variate α-stable density like the one of the underlying noise. In particular, for $\alpha = 2$, the stationary density is the bi-variate Gaussian distribution, while for $\alpha = 1$ it is the bi-variate Cauchy distribution.

For the quartic potential $V(x, y) = \frac{1}{4}(x^2 + y^2)^2$ and the uniform spectral measure the fractional diffusion equation (6) takes the form

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[(x^2 + y^2)xp\right] + \frac{\partial}{\partial y}\left[(x^2 + y^2)yp\right] - (-\Delta)^{\alpha/2}p. \tag{10}$$

In the Fourier space Eq. (10) takes the form

$$\frac{\partial \hat{p}}{\partial t} = k\frac{\partial^3 \hat{p}}{\partial k^3} + k\frac{\partial^3 \hat{p}}{\partial k \partial l^2} + l\frac{\partial^3 \hat{p}}{\partial k^2 \partial l} + l\frac{\partial^3 \hat{p}}{\partial l^3} - (k^2 + l^2)^{\alpha/2}\hat{p}. \tag{11}$$

The stationary density fulfills

$$-\frac{\hat{p}'}{z} + [\hat{p}'' - z^\alpha \hat{p}] + z\hat{p}''' = 0, \tag{12}$$

where $z = \sqrt{k^2 + l^2}$. Exact solutions can be constructed by numerical methods only, e.g. Langevin dynamics see Fig. 1.

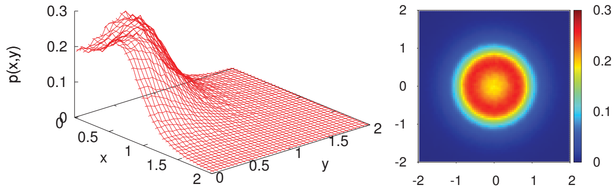

FIG. 1. Stationary states for the quartic potential $V(x, y) = \frac{1}{4}(x^2 + y^2)^2$ and α-stable noise with $\alpha = 1.0$, i.e. the Cauchy noise.

III. SUMMARY AND CONCLUSIONS

For the 2D harmonic potential stationary states reconstruct (up to rescaling) the noise distribution. In the limit of $\alpha = 2$ bi-variate α-stable densities converge to the bi-variate Gaussian distribution. Therefore, both types of bi-variate α-stable noises produces the same stationary state.

For the quartic potential stationary states are spherically symmetric and have local minima at the origin. With increasing value of the stability index α minima become shallower. Finally, in the limit of $\alpha = 2$ the Boltzmann-Gibbs distribution is reconstructed.

In general for single well potentials of $(x^2 + y^2)^{c/2}$ type (with $c \geqslant 2$) subject to action of bi-variate α-stable noises with uniform spectral measures stationary densities have power-law $(x^2 + y^2)^{-(c+\alpha)/2}$ asymptotics. Consequently, marginal densities have also power-law asymptotics with the exponent $-(c + \alpha - 1)$.

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