Modeling long-range dependent inverse cubic distributions by nonlinear stochastic differential equations

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I. INTRODUCTION

Many systems show large fluctuations of macroscopic quantities that follow non-Gaussian, heavy-tailed, power-law distributions with the power-law temporal correlations[1], scaling, and the fractal features[2,3]. The power-law distributions are often related both with the nonextensive statistical mechanics[4–6] and the power-law behavior of the power spectral density, i.e., $1/f^{\beta}$ noise "ambiguity" (see, e.g.,[3,6–8] and references herein).

One common way for describing all the above-mentioned forms of evolution is by means of the stochastic differential equations[7,9,10]. These nondeterministic equations of motion are used in many systems of interest, such as simulating the Brownian motion in statistical mechanics, field theory models, the financial systems, biology, and in many other areas.

One of the principal statistical features characterizing the activity in financial markets is the distribution of fluctuations of market indicators such as the indexes. Frequently heavy-tailed long-range distributions with characteristic power-law exponents are observable. Power laws appear for relevant financial fluctuations, such as fluctuations of number of trades, trading volume and price. The well-identified stylized fact is the so-called inverse cubic power-law of the cumulative distributions, which is relevant to the developed stock markets, to the commodity one, as well as to the most traded currency exchange rates. The exponents that characterize these power laws are similar for different types and sizes of markets, for different market trends and even for different countries—suggesting that a generic theoretical basis may inspire these phenomena[11–14].

Here we model the long-range dependent inverse cubic cumulative distributions by square multiplicative stochastic differential equations[7,10] and taking into account a transition from Stratonovich to Ito convention in noisy systems[15] according to Wong-Zakai theorem[16], with decrease of the driving noise correlation time when the market proceeds from turbulent to calm behavior.

II. STOCHASTIC DIFFERENTIAL EQUATIONS

We start from the squared stochastic differential equation (SDE)

$$dx = x^2 \circ_{\alpha} dW_t \tag{1}$$

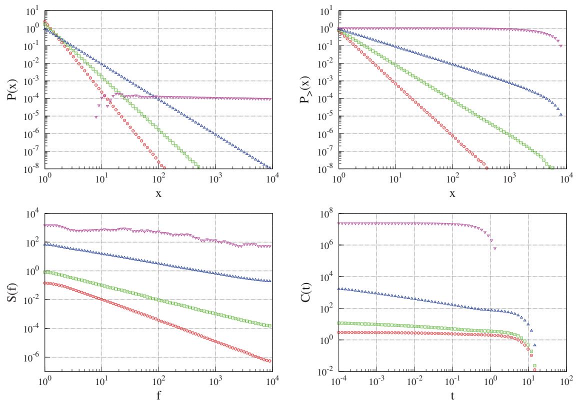

FIG. 1. The steady-state probability distribution function, cumulative distribution, power spectral density and autocorrelation function of the variable x generated by Eq. (3) for different parameters $\alpha = 0$; 1/4; 1/2 and 1 with restriction between $x_{\min} = 1$ and $x_{\max} = 10^4$.

where W_t is a Wiener process and α is the interpretation (convention) parameter, defining the α-dependent stochastic integral in Eq. (1),

$$\int_0^T f(x(t)) \circ_{\alpha} dW_t \equiv \lim_{N\to\infty} \sum_{n=0}^{N-1} f(x(t_n))\Delta W_{t_n}. \tag{2}$$

Here $t_n = \frac{n+\alpha}{N}T$, $0 \le \alpha \le 1$. Common choices of the parameter α are: (i) $\alpha = 0$, pre-point (Itô convention), (ii) $\alpha = 1/2$, mid-point (Stratonovich convention) and (iii) $\alpha = 1$, post-point (Hänggi-Klimontovich, kinetic or isothermal convention). More generally, the value of α may be variable, even coordinate x and/or the system parameters dependent quantity. Eq. (1) with $\alpha \neq 0$ may be transformed to Itô equation

$$dx = 2\alpha x^3 dt + x^2 dW_t \tag{3}$$

Eq. (3) is a particular case of the general Itô equations

$$dx = \left(\eta - \frac{\lambda}{2}\right) x^{2\eta-1} dt + x^{\eta} dW_t,\ \eta \neq 1, \tag{4}$$

yielding the power-law steady-state, $P_{ss}(x) \sim x^{-\lambda}$, distribution of the signal with the power-law spectrum, $S(f) \sim 1/f^{\beta}$, with the exponent

$$\beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}. \tag{5}$$

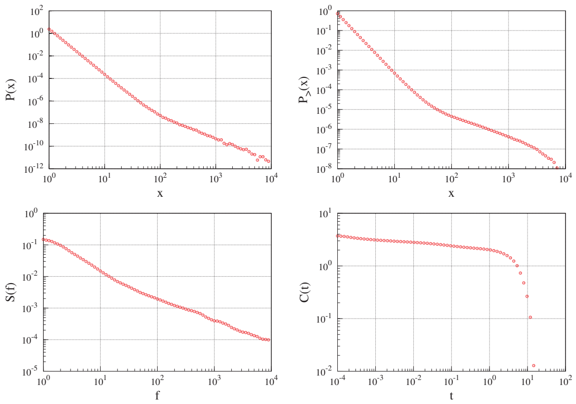

FIG. 2. The steady-state probability distribution function, cumulative distribution, power spectral density and autocorrelation function generated by Eqs. (6)–(9) with $x_c = 100$ and restriction between $x_{\min} = 1$ and $x_{\max} = 10^4$.

The relations between the parameters α and λ in Eqs. (3) and (4) for $\eta = 2$ are: $\lambda = 4(1-\alpha)$, $\alpha = 1-\lambda/4$. It should be noted that for the cumulative inverse cubic distribution $P_>(x) \sim x^{-3}$, i.e., $\lambda = 4$, according to Eq. (5) $\beta > 1$, for all $\eta > 1$ and, therefore, the modeled process is not long-range dependent. [Note that the definition of the long-range process corresponds the power-law autocorrelation function $C(t) \sim 1/t^\gamma$ with $0 < \gamma < 1$, which takes place for $0 < \beta < 1$ and $\gamma = 1 - \beta$.] Fig. 1 demonstrates statistics of solutions of Eq. (3) for different values of the parameter $\alpha = 0$; $1/4$; $1/2$ and 1, i.e., for $\lambda = 4$; 3; 2 and 0.

III. LONG-RANGE DEPENDENT INVERSE CUBIC DISTRIBUTION

For modelling of this phenomena we generalise Eqs. (1)–(4) with $\alpha(x)$-dependent parameter, e.g.,

$$\alpha(x) = \frac{1}{2}\left[1 - \exp\left\{-\left(\frac{x}{x_c}\right)^2\right\}\right], \tag{6}$$

where x_c is the process crossover parameter,

$$dx = 2\alpha(x)x^3 dt + x^2 dW_t \tag{7}$$

Eqs. (6) and (7) represents transition from Stratonovich to Itô convention with decreasing variable x and the driving noise correlation time for small x, according to Wong-Zakai theorem. The calculations are performed with the variable step of integration

$$\Delta t_k = \kappa^2 / x_k^2 \tag{8}$$

with $\kappa \ll 1$, yielding to the difference equation

$$x_{k+1} = x_k + 2\kappa^2\alpha(x_k)x_k + \kappa x_k \varepsilon_k. \tag{9}$$

Here ε_k is a set of uncorrelated normally distributed random variables with zero expectation and unit variance. Fig. 2 demonstrates results of the numerical calculations.

IV. CONCLUSION

Equations (6) and (7) with the variable dependent convention parameter $\alpha(x)$, according to Wong-Zakai theorem modeling decrease of the driving noise correlation time when the market proceeds from turbulent to calm behavior, may reproduce the long-range dependent inverse cubic phenomena.

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