Non-hermitian diffusion

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I. HISTORICAL INSPIRATION

In 1962 Dyson suggested an inspiring way to understand the joint probability distribution (jpd) of the eigenvalues of random matrices. In order to find the jpd, he was inventing an auxiliary dynamics undergoing in some fictitious "time", which, at the static limit, will lead to the stationary state (Gibbs state), representing the desired jpd. As he pointed[1], *"after considerable and fruitless efforts to develop a Newtonian theory of ensembles, we discovered that the correct procedure is quite different and much simpler. The x_i* [eigenvalues] *should be interpreted as positions of particles in Brownian motion.* The resulting stationary distributions (originally for hermitian or for unitary random matrices) were obtained as an effect of Ornstein-Uhlenbeck diffusion with a drift force coming from electrostatic-like repulsion of eigenvalues. The success of this description has contributed to multiple applications of random matrix models in practically all branches of science. The notion of "time" has evolved as well, so nowadays the "time" can be a physical dynamical parameter, representing either the real time or, e.g., the length of the mesoscopic wire, area of the string or external temperature. The idea of noisy walk of eigenvalues led also recently to such concepts as the study of determinantal processes, Loewner diffusion, fluctuations of non-intersecting interfaces in thermal equilibrium and the emergence of pre-shock spectral waves and universal scaling at the critical points of several random matrix models.

Three years after Dyson, Ginibre[2] has considered for the first time strictly non-hermitian random matrix modes, whose spectrum does not need to be confined either to real line (hermitian operators) or to unit circle (unitary operators), but can be located on the two-dimensional supports on the complex plane. Original motivation for the study of complex, random spectra was purely academic. Today, however, non-hermitian random operators play role in quantum information processing, in financial engineering (when lagged correlations are discussed) or in identifying clusters in social or biological networks using non-backtracking operators, to name just a few recent applications. Additionally, statistical properties of eigenvectors of non-hermitian operators contribute to understanding scattering problems in open chaotic cavities and random lasing.

II. MAIN RESULTS

In this contribution, following our recent work[3], we combine the original ideas of noisy random walk with the strict non-hermiticity of the operators, studying an evolution of Ginibre matrices whose elements undergo Brownian motion. The non-hermitian character of the Ginibre ensemble binds the dynamics of eigenvalues to the evolution of eigenvectors in a non-trivial way, leading to a system of coupled nonlinear equations resembling those for turbulent systems. We formulate a mathematical framework allowing simultaneous description of the flow of eigenvalues and eigenvectors, and we unravel a hidden dynamics as a function of new complex variable, which in a standard description is treated as a regulator only. We solve the evolution equations for large matrices and demonstrate that the non-analytic behavior of the Green's functions is associated with a shock wave stemming from a Burgers-like equation describing correlations of eigenvectors.

III. CONCLUSIONS AND OPEN PROBLEMS

We have proven that a consistent description of non-hermitian Gaussian ensemble requires the knowledge of the detailed dynamics of co-evolving eigenvalues and eigenvectors. Moreover, the dynamics of eigenvectors plays the superior role and leads directly to the inference of the spectral properties. This is dramatically different scenario comparing to the standard random matrix models, where the statistical properties of eigenvalues are of primary importance, and the properties of eigenvectors are basically trivial due to the their decoupling from the spectra. We conjecture that the discovered by us hidden dynamics of eigenvectors, that we have observed for the Ginibre ensemble, is a general feature of all non-hermitian random matrix models.

Our formalism could be exploited to expand the area of application of non-Hermitian random matrix ensembles within problems of growth, charged droplets in quantum Hall effect and gauge theory/geometry relations in string theory beyond the subclass of complex matrices represented by normal matrices.

One of the challenges is an explanation, why, despite being so different, Smoluchowski-Fokker-Planck equations for hermitian and non-hermitian random matrix models exhibit structural similarity to simple models of turbulence, where so-called Burgers equation plays the vital role, establishing the flow of the spectral density of eigenvalues in the case of the hermitian or unitary ensembles and the flow of certain eigenvector correlator in the case of non-hermitian ensembles.

We believe that our findings will contribute to understand several puzzles of non-hermitian dynamics, alike extreme sensitivity of spectra of non-hermitian systems to perturbations[4,5].

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