The spectral characteristics of steady-state Lévy flights in an infinitely deep rectangular potential well

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The anomalous diffusion in the form of Lévy flights is of permanent interest due to wide application in different areas of science[1–3]. At the same time, to explore this phenomenon, unlike the standard Brownian motion, one requests to apply the complex apparatus of fractional derivatives and state non-trivial boundary conditions. As a result, the steady-state probability density function of the particle coordinate can be found only for some simple potential profiles[4–6].

Investigation of the spectral properties of the steady-state Lévy flights in potentials with sufficiently steepness remains an unsolved problem. Here one can mention the exact result for the correlation time of Lévy flights in the symmetric quartic potential recently obtained in the article[7].

Our main goal is to find the spectral power density of the coordinates of the particles by diffusion in the form of the steady-state Lévy flights in an infinitely rectangular deep potential well

$$U(x) = \begin{cases} 0, & |x| < L, \\ \infty, & |x| > L. \end{cases} \tag{1}$$

It should be emphasized that the exact formula for the spectral density of the coordinate of Brownian particle moving in the potential (1) is given by[8]

$$S(\omega) = \frac{2D}{\omega^2}\left(1 - \frac{1}{L}\sqrt{\frac{D}{2\omega}} \cdot \frac{\sinh L\sqrt{2\omega/D} + \sin L\sqrt{2\omega/D}}{\cosh L\sqrt{2\omega/D} + \cos L\sqrt{2\omega/D}}\right), \tag{2}$$

where D is the diffusion coefficient.

We start from the following general operator formula for the correlation function $K[\tau]$ of a stationary Markovian process $x(t)$ previously obtained in the paper[9]

$$K[\tau] = \left\langle x\, e^{\hat{L}^{+}(x)\tau} x \right\rangle_{st}, \tag{3}$$

where $\hat{L}^{+}(x)$ is the adjoint kinetic operator and $\langle \ldots \rangle_{st}$ denotes averaging on the steady-state probability density function.

According to the Wiener-Khinchin theorem the spectral power density reads

$$S(\omega) = \int_{-\infty}^{\infty} K[\tau]\cos\omega\tau\, d\tau = 2\mathrm{Re}\left\{\tilde{K}[i\omega]\right\}, \tag{4}$$

where $\tilde{K}[p]$ is the Laplace transform of $K[\tau]$. From equation (4) we arrive at

$$\tilde{K}[p] = \left\langle x \frac{1}{p - \hat{L}^{+}(x)} x \right\rangle_{st}. \tag{5}$$

According to equation (5), one has to solve the following differential equation for the auxiliary function $\varphi(x)$

$$\hat{L}^{+}(x)\varphi(x) - p\varphi(x) = -x \tag{6}$$

and to find the average $\tilde{K}[p] = \langle x\varphi(x)\rangle_{st}$.

In particular, from equation (4) the correlation time can be calculated as

$$\tau_k = \frac{1}{\langle x, x\rangle}\int_0^{\infty} K[\tau]\, d\tau = \frac{S(0)}{2\langle x, x\rangle}, \tag{7}$$

where $\langle x, x\rangle$ is the variance of the particle coordinate.

Further we consider the anomalous diffusion in the form of Lévy flights in a potential $U(x)$ which is governed by the following Langevin equation for the particle coordinate $x(t)$

$$\frac{dx}{dt} = -\frac{dU(x)}{dx} + \xi_\alpha(t), \tag{8}$$

where α is the Lévy index ($0 < \alpha < 2$) and $\xi_\alpha(t)$ is the symmetric α-stable Lévy noise.

The corresponding fractional Fokker-Planck equation for the probability density function takes the form[3]

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(\frac{dU}{dx}P\right) + D_\alpha \frac{\partial^\alpha P}{\partial |x|^\alpha}. \tag{9}$$

For potential (1) the boundaries at $x = \pm L$ are impermeable for particles, i.e. $P(x,t) = 0$ at $|x| > L$. In the stationary case, according to the definition of the spatial fractional derivative, equation (9) transforms to

$$\int_{-L}^{L} \frac{P_{st}(z) - P_{st}(x)}{|x - z|^{1+\alpha}}\, dz = 0. \tag{10}$$

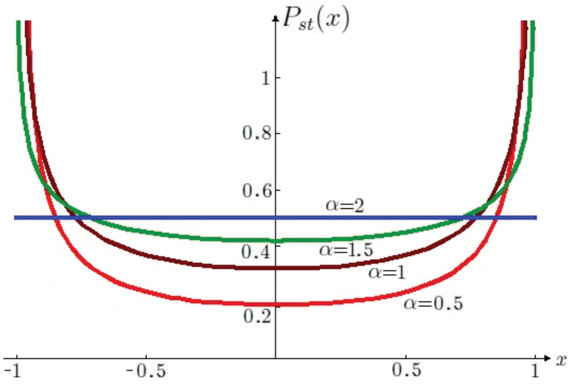

FIG. 1. Plots of stationary probability density for different values of the Lévy index α. The value of the parameter $L = 1$. The case $\alpha = 2$ corresponds to usual Brownian motion.

The solution of the integral equation (10) has been found in the article[10] and has the following form

$$P_{st}(x) = \frac{(2L)^{1-\alpha}\,\Gamma(\alpha)}{\Gamma^2(\alpha/2)(L^2 - x^2)^{1-\alpha/2}}, \qquad (11)$$

where $\Gamma(x)$ is the Gamma function.

The stationary probability density function (11) for different Lévy index α is shown in Fig. 1. It should be noted that the result (11) in the case $\alpha = 1$ can be derived from the steady-state probability distribution for smooth potential $U(x) = \dfrac{\gamma}{2m}\left(\dfrac{x}{L}\right)^{2m}$, previously obtained in article[6], in the limit $m \to \infty$.

Substituting the operator $\hat{L}^{+}(x)$ in equation (6), we need to solve the following integral equation for the function $\varphi(x)$

$$\int_{-L}^{L} \frac{\varphi(z) - \varphi(x)}{|x - z|^{1+\alpha}} dz - p\varphi(x) = -x. \qquad (12)$$

The spectral characteristics in the steady state for asymmetric Lévy flights in potential profile considered still remain an open problem. Moreover, the role of Lévy index α to get steady-state characteristics in general potential profiles is still unknown.

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