$1/f^\beta$ fluctuations from sequences of rectangular pulses

Vaidas Juknevičius, Bronislovas Kaulakys, and Julius Ruseckas
Institute of Theoretical Physics and Astronomy,
Vilnius University, A. Goštauto 12,
LT-01108 Vilnius, Lithuania
e-mail address: bronislovas.kaulakys@tfai.vu.lt

I. INTRODUCTION

Many processes observed in nature, technology and some other areas can be viewed as consisting of many separate events. Such events are localized in time in the sense that their contribution to the whole process is significant only in time intervals that are much shorter than the observation time of the whole process. The processes are usually measured as a change of some quantity y in time t that can be called a signal $y(t)$. Each discrete event that contributes to the process in question can therefore be represented by a *pulse* in the signal $y(t)$ with definite *time of occurrence* and *duration*[1].

Even though the underlying discreteness of the process may not be obvious due to the fact that often only a collective effect of a large number of contributing sources can be observed, the understanding of many phenomena still requires the insight into the discrete nature of entities creating the signal. Most notable, much researched, but still not completely solved problem is the $1/f$ noise in electronic devices[2].

We investigate stochastic signals that consist of rectangular pulses. The stochasticity here appears as a consequence of random variation of pulse duration and timing of the pulse occurrence. The power-spectral densities (PSD's) of such sequences of pulses under appropriate conditions have a power-law shape[1]. Since the power-law-shaped PSD's are ubiquitous in natural phenomena, technology and even quantitative social sciences, the investigation of the conditions under which pulse sequences exhibit such PSD's can be useful for the better understanding of a wide range of phenomena.

In this contribution we present the conditions under which the PSD $S(f)$ of the sequence of pulses obtains power-law shape for small frequencies $S(f) \sim 1/f^\beta$ and how the spectral power β depends on the statistical parameters of pulse timing. We also present a model of charge carrier trapping in disordered materials that satisfies the conditions for producing $1/f$ noise.

II. OVERLAPPING AND NON-OVERLAPPING PULSES

Let us consider a process that consists of discrete events and can be represented by a signal $y(t)$ – a sequence of pulses. $y(t)$ can be expressed as a sum of individual pulses $x_k(t)$ shifted in time:

$$y(t) = \sum_k x_k(t - t_k) \tag{1}$$

Here $x_k(t)$ is the shape of the k-th pulse and t_k - the time of its occurrence. The shape of each pulse $x_k(t)$ is described by a set of parameters (for example, the amplitude, duration etc.) which can obtain random values.

We investigate a simple case of stationary sequences of rectangular pulses of constant amplitude whose durations $\{\tau_k\}$ are independent identically distributed (i.i.d.) random variables.

The timing of pulses can be defined either by the time interval $\theta_k = t_{k+1} - t_k$ between the occurrence of successive k-th and $(k+1)$-st pulses or by the time interval $\delta_k = t_{k+1} - t_k - \tau_k$ between the end of the k-th pulse (at time $t_k + \tau_k$) and the start of the $(k+1)$-st pulse (at time $t_{k+1} > t_k + \tau_k$). We will call δ_k the *gap* between successive pulses. We assume that one of these two quantities – either θ_k or δ_k – together with the pulse duration τ_k are i.i.d., and thus there are two distinct possibilities for the construction of the pulse sequences in this case: *(possibly) overlapping pulses* where the duration and timing of pulses are defined by two independent quantities τ_k and θ_k, and *non-overlapping* pulses where the $(k+1)$-st pulse begins only after the k-th pulse ends, and therefore the two independent quantities describing the pulse duration and timing are τ_k and δ_k.

A schematic representation of a signal described above in the case of rectangular pulses with constant amplitude a is given in Fig. 1.

III. SPECTRAL PROPERTIES

The one-sided power-spectral density (PSD) of the signal $y(t)$ is defined as follows:

$$S(f) = \lim_{T\to\infty} \left\langle \frac{2}{T} \left| \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\; y(t)\, e^{-i2\pi f t} \right|^2 \right\rangle \tag{2}$$

Here T is the observation time, $T = t_\mathrm{f} - t_\mathrm{i}$.

Under the assumptions of stationarity and ergodicity of $y(t)$ (1) with pulses $x_k(t)$ sufficiently localized in time, one gets the following PSD's for rectangular pulses with *exponentially distributed durations* τ_k:

$$S(f) = \frac{4\,\bar{\nu}\,a^2\bar{\tau}^2}{1 + (2\pi f\bar{\tau})^2} \left[\mathfrak{Re}\left\{ \frac{\chi_\theta(f)}{1 - \chi_\theta(f)} \right\} + 1 \right] \tag{3}$$

for possibly overlapping pulses and

$$S(f) = \frac{4\,\bar{\nu}\,a^2\bar{\tau}^2}{1 + (2\pi f\bar{\tau})^2} \mathfrak{Re}\left\{ \frac{1 - \mathrm{i}2\pi f\bar{\tau}(1 - \chi_\delta(f))}{(1 - \chi_\delta(f)) - \mathrm{i}2\pi f\bar{\tau}} \right\} \tag{4}$$

for non-overlapping pulses (this case is often called *random telegraph noise*). Here $\mathfrak{Re}$ denotes the real part, $\chi_\theta(f)$ and $\chi_\delta(f)$ are the characteristic functions of θ_k the and δ_k, respectively. $\bar{\nu}$ is the *average rate of pulse occurrence* which is $\bar{\nu} = \bar{\theta}^{-1}$ and $\bar{\nu} = (\bar{\theta} + \bar{\delta})^{-1}$ for the respective cases (3) and (4), and $\bar{\tau}$, $\bar{\theta}$ and $\bar{\delta}$ are the averages of the quantities τ_k, θ_k and δ_k, respectively.

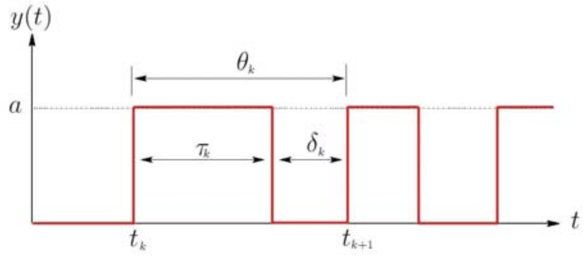

FIG. 1. A schematic representation of a signal, consisting of rectangular pulses with constant amplitude a. The three quantities that define pulse timing (only two of them independent): pulse *duration* τ_k, *gap* between successive pulses δ_k and *interpulse time* (time between the occurrence of successive pulses) θ_k.

IV. POWER-LAW SPECTRA

Analytical and numerical calculations suggest that, for sequences of rectangular pulses (1) with exponentially distributed pulse durations, the *power-law* PSD with power β for small frequencies

$$S(f) \sim \frac{1}{f^\beta} \tag{5}$$

can be obtained in both cases (3) and (4) when either the times between the occurrence of successive pulses θ_k or the gap durations δ_k between pulses are power-law distributed with the power $\gamma > 1$ or, alternatively, $\gamma = 1 + \alpha$ with $\alpha > 0$:

$$p(\vartheta) \sim \vartheta^{-(1+\alpha)} \ , \quad \vartheta_{\min} \leq \vartheta < \vartheta_{\max} \ , \tag{6}$$

Here $p(\vartheta)$ denotes the probability density of the quantity ϑ which can be one of the quantities θ_k or δ_k and the bounds of the power-law region $\vartheta_{\max} \gg \vartheta_{\min}$.

The additional condition for the occurrence of the power-law PSD (5) is that the average $\bar{\tau}$ of the exponentially distributed pulse duration τ_k must be greater than the average inter-pulse time $\bar{\vartheta}$, that is $\bar{\tau} \gg \bar{\vartheta}$.[3] For $\alpha > 1$ we can get the finite average value $\bar{\vartheta} = \frac{\alpha}{\alpha-1}\vartheta_{\min}$ for $\vartheta_{\max} \to \infty$. However, in order for $\bar{\vartheta}$ to be finite in the case $\alpha \leq 1$, $\vartheta_{\max}$ must be finite.

If the above conditions are fulfilled, then we get the power-law PSD (5) for the frequencies between $f_{\min} \approx \vartheta_{\max}^{-1}$ and $f_{\max} \approx \vartheta_{\min}^{-1}$ where the power β depends on α as follows:

$$\beta(\alpha) \approx \begin{cases} \alpha & \alpha < 1 \\ 2 - \alpha & 1 \leq \alpha < 2 \\ 0 & \alpha \geq 2 \end{cases} \tag{7}$$

We see that the $1/f$ noise (β=1 in (5)) is obtained when $\alpha = 1$, i.e., the distribution of the inter-pulse time ϑ (either θ_k or δ_k) has a power-law distribution $p(\vartheta) \sim \vartheta^{-2}$ for a wide range of ϑ. Such case is shown in Fig. 2.

V. CHARGE CARRIER TRAPPING

The results presented above have been applied to the model describing current fluctuations in defective materials due to the charge carrier trapping[4].

The model states that a charge carrier moving through some disordered material is successively trapped in and released from trapping centers with widely distributed release rates. For the appropriately chosen distribution of the release rates, the resulting current corresponds to the signal of non-overlapping pulses with power-law distribution of gaps between pulses with the power -2, resulting in the $1/f$ PSD in a wide range of frequencies.

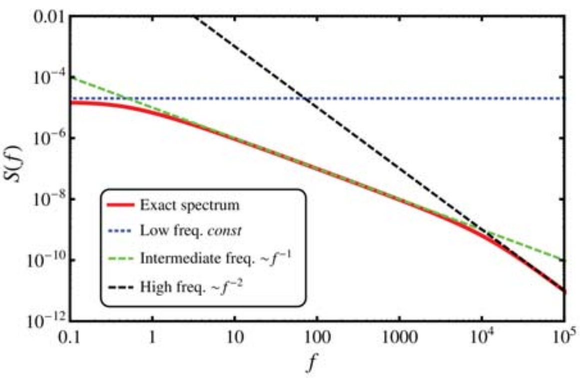

FIG. 2. The PSD $S(f)$ resulting from the model of charged particle trapping. The thick red line shows the exactly calculated spectrum, dashed and dotted lines are approximations for different frequency regions (see legend).

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