Brownian motion and weak ergodicity breaking

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I. INTRODUCTION

Non-ergodicity observed in single-particle tracking experiments is usually modeled by transient trapping rather than spatial disorder. I will talk about our models[1] of a particle undergoing Brownian motion in a medium with inhomogeneous random diffusivities, but no traps. For some values of model parameters, we find that the mean squared displacement displays subdiffusion due to non-ergodicity for both annealed and quenched disorder. This is significant because Brownian motion itself is the prototypical example of ergodic, normal, diffusion. I will also discuss recent results on extensions of these models to quenched 2-d disorder.

The significance of this work is best understood in the context of unanswered questions in the study of of naturally occurring stochastic transport. Consider, for example, receptors (molecules) diffusing laterally in the cell membrane. In this setting, these questions include, “To what extent do tracking data point to ergodic or non-ergodic processes”, and, “What are the physical/structural mechanisms behind these processes?”. In the case that non-ergodic processes are observed, basic characteristics of traps, such as their size or physical origin are not known. Furthermore, the present work shows that it is not even necessary to assume the presence of traps. We have, in fact, found that there are systems for which inhomogeneity provides a quantitatively more accurate description than than traps[2].

In the talk, I will focus mainly on the theoretical story, touching only briefly on the application to real systems.

II. MODEL

In our models of diffusion in inhomogeneous media, one may assume annealed or quenched disorder. For concreteness, we introduce here a particular annealed model. We consider a particle that undergoes Brownian motion with a random diffusivity for a random time. Then, new, independent random diffusivities and times are chosen and the particle again undergoes Brownian motion. This process is repeated. The asymptotic behavior, in particular the mean squared displacement, shows either ordinary or anomalous behavior with weak-ergodicity breaking depending on the model parameters.

More precisely, consider sequences of random variables, the diffusivities $\{D_j\}$ and the transit times $\{\tau_j\}$. The elements of $\{D_j\}$ are identically distributed as are the elements of $\{\tau_j\}$ All pairs are independently distributed with the exception of pairs (D_j, τ_j), for all j. We assume the (common) probability density function (PDF) for D_j has the form

$$P_D(D) \sim D^{\sigma-1} \qquad \text{with } \sigma > 0, \tag{1}$$

for small D. Furthermore, we require that the PDF for transit times τ conditioned on D, $P_\tau(\tau|D)$, has mean

$$\mathrm{E}\left[\tau|D\right] = D^{-\gamma} \qquad \text{with } -\infty < \gamma < \infty, \tag{2}$$

with all moments $P_\tau(\tau|D)$ being finite. In this simple case, the two parameters σ and γ characterize the model completely.

We analyzed this model in a generalized continuous time random walk framework using Fourier-Laplace transforms. In particular, the asymptotic ensemble averaged mean-squared displacement (MSD) is

$$\left\langle x^2(t)\right\rangle \sim t^\beta \qquad \text{with } 0 \le \beta \le 1, \tag{3}$$

and β taking values depending on the model parameters as shown in Tab. (1). Exponents for the model in one-dimension when the diffusivities are quenched (fixed in space for the duration of the walk) are also shown.

	(0) $\gamma < \sigma$	**(I)** $\sigma < \gamma < \sigma + 1$	**(II)** $\sigma + 1 < \gamma$
Annealed	1	σ/γ	$1 - 1/\gamma$
Quenched 1d	1	$2\sigma/(\sigma+\gamma)$	Unknown

TAB. 1. Ensemble averaged MSD exponent β in (3) for the annealed model, and the one-dimensional quenched model, as a function of σ and γ defined in (1) and (2). The exponent β for the 1D quenched case in region II is unknown at present.

By itself, the anomalous exponent in (3) is not enough to demonstrate weak ergodicity breaking. However with some additional assumptions, which are supported by Monte Carlo studies, one finds that the result of performing both the ensemble averaged and the time averaged MSD has the asymptotic form

$$\overline{\langle x^2(t)\rangle}_T \sim T^{\beta-1} t \qquad \text{for } t \ll T \tag{4}$$

where T is the observation time over which time averages of functions of the lag time t are performed. The most significant feature of (4) is that it is not equal to the ensemble averaged quantity in (3). That is, the motion is not ergodic. Furthermore, as is the case with the standard continuous time random walk, for short lag times t, the particle shows ordinary diffusion.

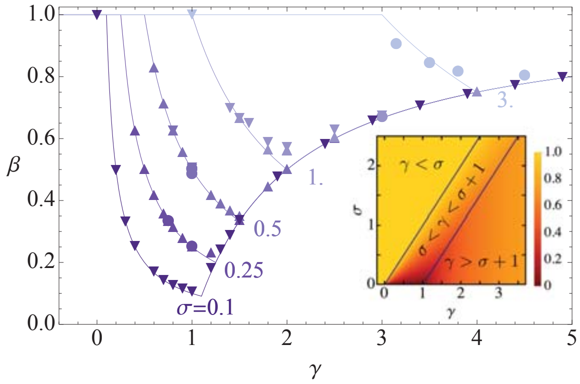

FIG. 1: Exponent β in (3) for annealed models. Lines are analytic results as in Tab. (1)., for various values of σ indicated in the figure. Symbols are numerical simulations. Numerical estimates of β are extracted by fitting Monte Carlo simulations of the ensemble average of annealed models (▼ and ▲) to (3), and simulations of the combined time-ensemble average (circles) to (4). The inset shows a density plot of β vs. both γ and σ.

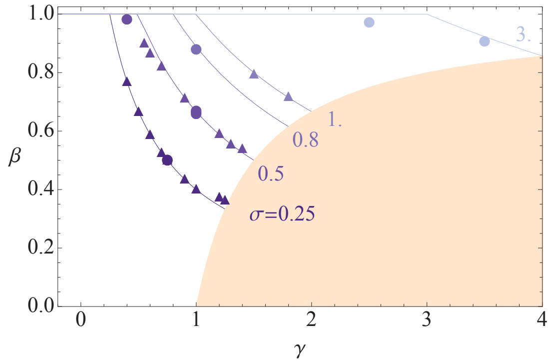

FIG. 2. Exponent β in (3) for the 1d quenched model. Lines as in Fig. 1. Symbols are exponents extracted from numerical simulations of the ensemble MSD (▲) and time-ensemble MSD (circles). Lines and symbols vary from dark to light with increasing σ. Shading indicates region (**II**), where the exponent is at present unknown.

The analytically determined exponents and Monte Carlo estimates for the annealed model are plotted in Fig. (1). The lines in the figure were drawn by fixing σ at a sequence of values and varying γ along the coordinate axis, computing the analytic expression for β. The points correspond to Monte Carlo estimates of β computed for various values of σ and γ. An analogous plot for the 1-d annealed case is shown in Fig. (2) We are currently studying higher-dimensional quenched models. For many situations (specifically, when spatial correlations are not important), we expect to recover the annealed exponents.

To reiterate, the significance of this work in the context of this conference is that it provides a clear example of the fact the the characterization of the possible subdiffusive processes arising from disorder is far from closed. In particular, if the distribution of diffusivities is anomalous, even ordinary Brownian motion can exhibit anomalous statistics.

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[1] Non-ergodic subdiffusion from Brownian motion in an inhomogeneous medium, P. Massignan, C. Manzo, J. A. Torreno-Pina, M. F. Garca-Parajo, M. Lewenstein, and G. J. Lapeyre Jr Phys. Rev. Lett. **112**, 150603 (2014).

[2] Weak ergodicity breaking of receptor motion in living cells stemming from random diffusivity, C Manzo, J.A. Torreno-Pina, P. Massignan, G.J. Lapeyre Jr, M. Lewenstein, and M.F. Garcia-Parajo Accepted for publication in Phys. Rev. X., arxiv:1407.2552