Cascade Amplification of Fluctuations

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I. INTRODUCTION

We consider a dynamical system which has a stable attractor and which is perturbed by an additive noise. Under some quite typical conditions, the fluctuations from the attractor are intermittent and have a probability distribution with power-law tails. We show that this results from a cascade of amplification of fluctuations due to transient periods of instability.

II. POWER-LAW FLUCTUATIONS

In a one-dimensional example the equation of motion might be

$$\dot{x} = v(x,t) + \sqrt{2D}\eta(t) \tag{1}$$

where $\eta(t)$ is a white noise signal (defined by (2) below) and D is the diffusion coefficient of the corresponding Brownian motion. We shall be primarily concerned with the case where D is small. The underlying system (without the noise term) is taken to be stable in the sense that its Lyapunov exponent λ is negative, implying that nearby trajectories converge. When a small noise term is added to the equation of motion, the trajectories do not reach the attractor of the underlying deterministic system. The separation of two trajectories, Δx, can be characterised by its probability density $P_{\Delta x}$. (Several stochastic variables are introduced here: we use P_X to denote the probability density function for a quantity X, and $\langle X\rangle$ to denote its expectation value). It might be expected that $P_{\Delta x}$ would be well-approximated by a Gaussian distribution, and for a generic class of models this expectation is correct. An example is the case where the underlying dynamical system is $\dot{x} = \lambda x$ (with $\lambda < 0$), which approaches the attractor $x = 0$. If this equation of motion is replaced by $\dot{x} = \lambda x + \sqrt{2D}\eta(t)$, where $\eta(t)$ is white noise, with statistics

$$\langle \eta(t)\rangle = 0\ , \qquad \langle \eta(t)\eta(t')\rangle = \delta(t-t') \tag{2}$$

these equations describe an Ornstein-Uhlenbeck[1] process. The deviations from the fixed point have a Gaussian distribution in the limit as $t \to \infty$:

$$P_x = \sqrt{\frac{\lambda}{2\pi D}}\exp\left[-\frac{|\lambda| x^2}{2D}\right]\ . \tag{3}$$

provided $\lambda < 0$. The deviation Δx of two trajectories from each other is is also Gaussian distributed, with double the variance. We argue, however, that there is also a generic class of models for which the distribution $P_{\Delta x}(\Delta x)$ of separations of trajectories has power-law tails:

$$P_{\Delta x} \sim |\Delta x|^{-1+\alpha} \tag{4}$$

when Δ is large compared to $\sqrt{D/|\lambda|}$, (and we must have $\alpha > 0$ if this distribution is to be normalisable). The large excursions of $\Delta x(t)$ which are the origin of the power-law tails are a form of intermittency. The intermittency of $\Delta x(t)$ is illustrated in figure 1 for a model of colloidal particles in a turbulent flow. Figure 2 shows evidence that Δx has a power-law distribution in the same model, with an exponent which is independent of D. The power-law distribution has both an upper and a lower cutoff scale, and the lower cutoff scale decreases as $D \to 0$.

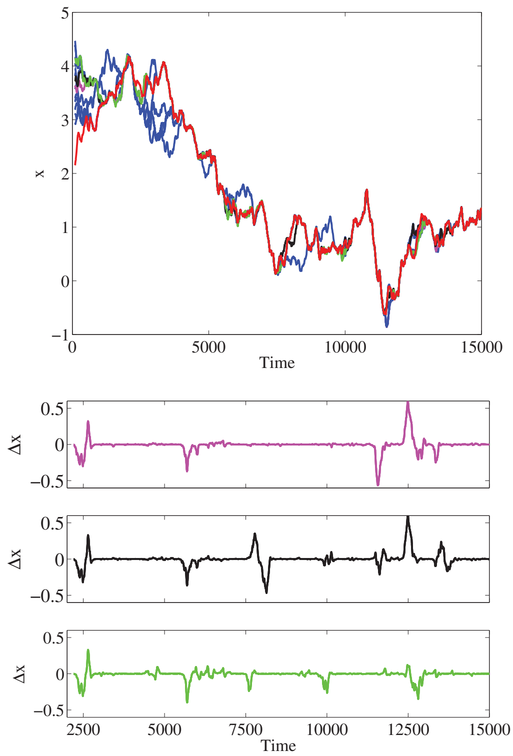

FIG. 1. **a** Set of trajectories of a model of colloidal particles in suspension. Different trajectories separate and recombine. **b** Intermittent separation of pairs of trajectories $\Delta x(t)$.

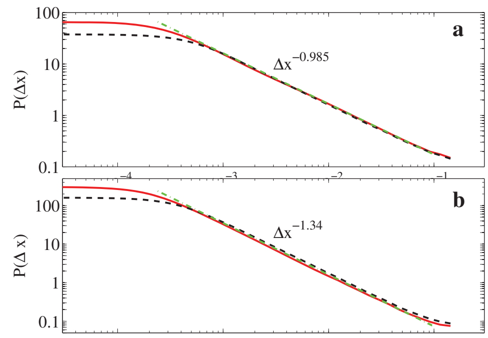

FIG. 2. The probability distribution $P_{\Delta x}$ for a model system, for two values of D ($D = 3\times10^{-10}$, full curve, and $D = 10^{-9}$, dashed curves, showing fits by a power law $|\Delta x|^{-(\alpha+1)}$.

In this talk we explain the origin of this intermittency of show how it can be quantified by making an analytical theory of the exponent α. The intermittency which is considered here is different from that which is discussed in most studies[2–5], where intermittency arises because a system is just above a threshold of instability. The systems considered in this work are driven by external noise, and we are concerned with what might be termed *sub-critical intermittency*, which is observed when the underlying system converges to an attractor. The intermittency effect discussed in this paper is only observed in non-autonomous systems, where the linearised dynamics in the vicinity of the attractor is fluctuating.

Beyond the point at which the underlying system becomes unstable, that is, when the Lyapunov exponent λ is greater than zero. The system ceases to have a point attractor, and instead it has a strange attractor, where phase points cluster on a fractal measure[5]. We argue that as λ approaches zero from below, the exponent α in (4) approaches zero from above. When $\lambda > 0$, the two-point correlation function of the strange attractor, $g(\Delta x)$, has a power-law dependence: $g(\Delta x) \sim |\Delta x|^{D_2-1}$ where D_2 is the correlation dimension of the strange attractor[6]. The analogy with (4) suggests that the exponents are related by

$$\alpha = -D_2 \tag{5}$$

so that normalisable distributions of fluctuations correspond to negative values of D_2. Equation (4) therefore gives a clear physical meaning to a negative fractal dimension.

III. EXPLANATION

The talk will present an explanation of the effect, expanding upon the summary below.

Consider the linearisation of Eq. (1) to give the separation between two nearby trajectories:

$$\delta\dot{x} = Z(t)\delta x + 2\sqrt{D}\eta(t)\ , \qquad Z(t) = \frac{\partial v}{\partial x}(x(t),t). \tag{6}$$

Note that when $D = 0$, $Z(t)$ is the logarithmic derivative of the separation $\delta x(t)$, and we can think of $Z(t)$ as being an *instantaneous Lyapunov exponent*. In the case of autonomous systems with an attractor, the attractor must be a fixed point in phase space, and $Z(t)$ approaches a constant $\lambda < 0$ as $t \to \infty$. In this case the fluctuations are described by an OU process and the distribution $P_{\Delta x}$ is Gaussian. In cases where the dynamical system is non-autonomous, $Z(t)$ need not approach a constant value. If the external driving is a stationary stochastic process, $Z(t)$ is a fluctuating quantity with stationary statistics. The origin of the power-law tails described by (4) is that the fluctuations are amplified during periods when $Z(t) > 0$. This noise amplification is independent of the initial amplitude, because the fluctuating quantity $Z(t)$ acts multiplicatively in Eq. (6). This leads to a stochastic cascade amplification process, whereby large amplitude fluctuations are built up by a succession of periods where $Z(t) > 0$. The power-law tail in the fluctuation distribution arises whenever $Z(t)$ is positive for some intervals of time, however short.

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