1/f noise arising from time-subordinated Langevin equations

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I. INTRODUCTION

Many complex systems exhibit large fluctuations of macroscopic quantities having non-Gaussian power law distributions as well as power law temporal correlations and scaling.[1] The power law distributions, scaling, self-similarity and fractality can be related to the power law behavior of the power spectral density (PSD), which is one of the most important characteristics of a signal. Signals having the PSD at low frequencies f of the form $S(f) \sim 1/f^{\beta}$ with β close to 1 are commonly referred to as "$1/f$ noise". Power-law distributions of spectra of signals with $0.5 < \beta < 1.5$, as well as scaling behavior are ubiquitous in physics and in many other fields. For recent reviews see 2–4. Despite the numerous models and theories proposed since its discovery 90 years ago, the subject of $1/f$ noise remains open for new discoveries. Most models and theories of $1/f$ noise are not universal due to the usage of assumptions specific to the problem under consideration.

Often $1/f$ noise is modeled as the superposition of Lorentzian spectra with a wide range distribution of relaxation times.[5] A class of the models of $1/f$ noise especially relevant for understanding of complex systems involves the self-organized criticality.[6] Yet another model of $1/f$ noise has been proposed by Kaulakys:[7,8] it has been shown that the origin of $1/f$ noise in a signal consisting of pulses may be a Brownian motion of the inter-pulse time. The nonlinear stochastic differential equations (SDEs) generating signals with $1/f$ noise has been obtained starting from this point process model of $1/f$ noise[9,10]. Such nonlinear SDEs have been used to describe signals in socio-economical systems[11,12].

In this contribution we generalize the mechanism leading to $1/f$ noise in the signals consisting of a sequence of pulses. Instead of a sequence of pulses we start from an SDE describing a Brownian motion in an external potential. We construct a new equation by interpreting the time in the SDE as an internal parameter and adding an additional equation relating the physical time to the internal time. We show that relation between the internal time and the physical time that depends on the size of the signal can lead to $1/f$ noise in a wide interval of frequencies.

II. $1/f$ NOISE AND DIFFUSION IN NON-HOMOGENEOUS MEDIA

Impurities and regular structures in the medium results in a transport of variable speed, the particle may be trapped for some time or accelerated. Non-homogeneous systems are characterized not only by subdiffusion related to traps, but also enhanced diffusion can arise as a result of the disorder.[13] The dynamics of such a system is described by the continuous time random walk (CTRW) theory. In an equivalent description the dynamics is Markovian and governed by a Langevin equation in an auxiliary, operational time instead of the physical time. This Markovian process is subordinated to the process yielding the physical time.

Since the trap properties should reflect the structure of the medium, a description of the transport should take into account that the waiting time explicitly depends on the position. Here we consider the situation when the small increments of the physical time are deterministic and proportional to the increments of the internal time. The coefficient of proportionality is a function $g(x)$ of a particle position. This function models the position of structures responsible for either trapping or accelerating the particle. Thus, we start from the following set of equations:

$$dx_{\tau} = F(x_{\tau})d\tau + dW_{\tau}\,, \tag{1}$$
$$dt_{\tau} = g(x_{\tau})d\tau\,. \tag{2}$$

Here τ is an internal, operational time and t is the physical time; $F(x)$ is an external force affecting the particle and W_{τ} is a standard Wiener process.

Writing the Fokker-Planck equation for the two-dimensional density $P(x,t)$ corresponding to (1), (2) and changing the variable t to a variable τ one can reduce the system of equations (1), (2) to a single equation in physical time with a multiplicative noise

$$dx_t = \frac{F(x_t)}{g(x_t)}dt + \frac{1}{\sqrt{g(x_t)}}dW_t\,. \tag{3}$$

There is a similarity to the signal consisting of pulses, where the internal time is just the pulse number. In order to obtain $1/f$ noise similarly as in a signal consisting of pulses we choose the function $g(x)$ as a power-law function of x: $g(x) \sim x^{-2\eta}$.

For example, if we start from a simple Brownian motion $dx_{\tau} = dW_{\tau}$ restricted to a interval between $x_{\min}$ and $x_{\max}$ and take $g(x) = x^{-2\eta}$ then the resulting equation

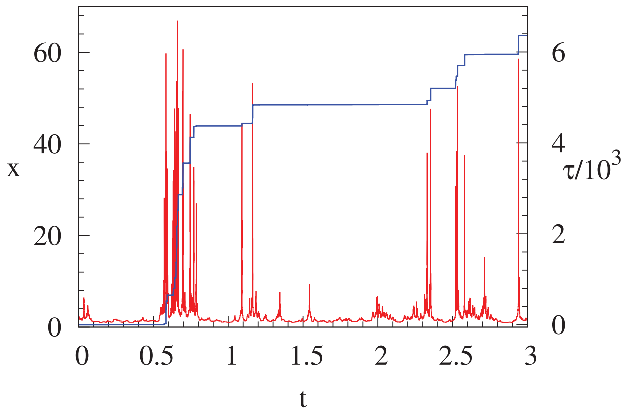

FIG. 1. Signal generated by equation (5) with the parameters $\eta = 5/2$ and $\lambda = 3$ (red line) together with the corresponding internal time (blue line)

in the physical time is $dx_t = x_t^{\eta} dW_t$. In more general situation the initial equation can have a position-dependent force. If we take the equation describing a Bessel process

$$dx_\tau = \left(\eta - \frac{\lambda}{2}\right) \frac{1}{x_\tau} d\tau + dW_\tau \tag{4}$$

then the resulting equation in the physical time becomes

$$dx_t = \left(\eta - \frac{\lambda}{2}\right) x_t^{2\eta-1} dt + x_t^{\eta} dW_t \tag{5}$$

This equation is the same as the nonlinear SDE generating signals with $1/f^{\beta}$ spectrum[9,10]. As has been shown,[14] the reason for the appearance of $1/f$ spectrum is the scaling properties of the signal: the change of the magnitude of the variable $x \to ax$ is equivalent to the change of the time scale $t \to a^{2(\eta-1)}t$.

Equation (4) together with $dt_\tau = x_\tau^{-2\eta} d\tau$ suggest an efficient way of solving the non-linear SDE (5). Discretizing the internal time τ with the step $\Delta\tau$ and using the Euler-Marujama approximation for the SDE (4) we get

$$x_{k+1} = x_k + \left(\eta - \frac{\nu}{2}\right) \frac{1}{x_k} \Delta\tau + \sqrt{\Delta\tau} \varepsilon_k \,, \tag{6}$$

$$t_{k+1} = t_k + \frac{\Delta\tau}{x_k^{2\eta}} \tag{7}$$

Here ε_k are normally distributed uncorrelated random variables.

An example of a signal generated by equation (5) together with the internal time is shown in Fig. 1. We see that internal time increases rapidly when the signal acquires large values. The corresponding spectrum is shown in Fig. 2. The numerical solution confirms a presence of a wide region where the spectrum behaves as $1/f$.

In summary, we have demonstrated that the Brownian motion in non-homogeneous medium can result in $1/f$ noise when the internal time and the physical time are related via power-law function of the position. We expect that the present model can be useful for explaining $1/f$ noise in complex systems.

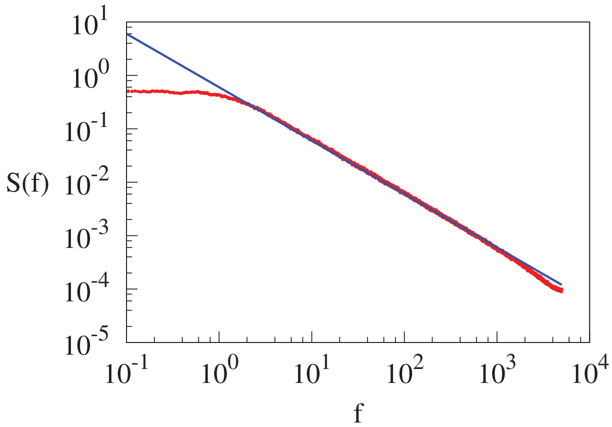

FIG. 2. Spectrum of the signal generated by equation (5) with the parameters $\eta = 5/2$ and $\lambda = 3$ (red curve). Blue line shows the slope $1/f$

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